Integration By Parts

Arianne Reidinger

The purpose of this application is to show how the matrix of a linear transformation may be used to calculate antiderivatives usually found by integration by parts.

- **Introduction**

Let's begin by taking the exponential function and applying the antiderivative to find a general rule for computing the integral with matrices. The basis B = {t^2e^t , t e^t , e^t } is a linearly independent set and is spans the matrix V. Making D the differentiation operator for all functions f in V, we can calculate the matrix for D relative to B. The symbol for this operation is denoted as [D]b. Therefore the equation that results is:

[D]b = { [D(t^2e^t)]b [D(t e^t)]b [D(e^t)b] }

We know the that [D]b = t^2e^t + 2t e^t ; t e^t + e^t ; e^t respectively. By constructing a 3x3 matrix of the coefficients of each equation, the matrix can then be used to differentiate any member of V.

$$[D]b = \begin{matrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$$

We can now use this method to find the antiderivative of a function. The important thing to notice is that [D]b is invertible because it's determinant is non-zero. D is then also an invertible linear transformation on V. The inverse of [D]b is the B-matrix of D-inverse. What this means is that the inverse of [D]b is the B-matrix for the antidifferentiation operator on V.

$$[D]b^{-1} = \begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{matrix}$$

This matrix will serve to be the antidifferentiation matrix in which we use for all functions including exponentials. To find the antiderivative of a function, multiply the antidifferentiation matrix with the coordinate vector of the function relative to the basis. It is that simple!

- **Application**

Here are some simple applications to illustrate using the D-inverse to find the antiderivative of an exponential function.

1. $\int t\, e^{t}\, dt$

First, determine what the coordinate vector is of this exponential function relative to the basis B. The coordinate vector is: $[\ t\ e^{t}\ ]b$ = (0,1,0). Next multiply by the antidifferentiation matrix:

$$\begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{matrix} \quad \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} = \begin{matrix} 1 \\ 1 \\ -1 \end{matrix}$$

The antiderivative of ∫ t e^t dt in the vector space of V is $t^2e^t + t\,e^t - e^t$. This answer can be verified by actually performing the integration by parts.

2. Find the antiderivative ∫ $t^3 e^t$ dt.

Let B = { $t^3 e^t$, t^2e^t , t e^t , e^t } and let V be in the vector space of the functions spanned by the functions in B. First, find the matrix [D]b for the differentiation operator D.

[D]b = { [D($t^3 e^t$)]b [D(t^2e^t)]b [D(t e^t)]b [D(e^t)b] }

We know the that [D]b = $3t^2e^t + t^3 e^t$; $t^2e^t + 2t\,e^t$; $t\,e^t + e^t$; e^t respectively. By constructing a 4x4 matrix of the coefficients of each equation, the matrix can then be used to differentiate any member of V.

$$[D]b = \begin{matrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}$$

The determinate of [D]b is non-zero which means D is invertible and has an inverse. Compute the inverse of D using a form of technology or by the method learned in section 2.2 of our textbook.

$$[D]b^{-1} = \begin{matrix} 1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{matrix}$$

The coordinate vector of $t^3 e^t$ is (1, 0, 0, 0). Multiply the antidifferentiation matrix by the coordinate vector and the result is the antiderivative.

$$\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & & 1 \\ -3 & 0 & 0 & 0 & 0 & = & -3 \\ 0 & -2 & 1 & 0 & 0 & & 0 \\ 0 & 2 & -1 & 0 & 0 & & 1 \end{array}$$

$\int t^3 e^t \, dt = t^3 e^t - 3 t^2 e^t + e^t$ which can be verified through integration by parts.

- **Conclusion**

In conclusion, the matrix of a linear transformation can be used to find the derivative and antiderivative of a function by choosing a basis that is respective of the matrix representation. The main idea that makes this antidifferentiation possible is the fact that the differentiation operator D is invertible. In order for this method of applying a matrix of a linear transformation to solve an integral that would normally be solved by integration by parts, you need to have a basis with respect to which the differentiation operator is invertible. The method illustrated above applies only to functions containing exponentials. It would be interesting to see what other matrices can be used to solve other functions.

- **Works Cited**

Johnson, Lee W., Introduction to Linear Algebra. New York: Addison-Wesley, 1999

Lay, David C., Linear Algebra and its Applications. New York: Addison-Wesley, 2000

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