NON-EUCLIDEAN GEOMETRY

FROM PARALLEL POSTULATE TO MODELS

GREEK GEOMETRY

- Greek Geometry was the first example of a deductive system with axioms, theorems, and proofs.

- Greek Geometry was thought of as an idealized model of the real world.

- Euclid (c. 330-275 BC) was the great expositor of Greek mathematics who brought together the work of generations in a book for the ages.

Euclid as Cultural Icon

Euclidean geometry was considered the apex of intellectual achievement for about 2000 years. It was the standard of excellence and model for math and science.

Euclid's text was used heavily through the nineteenth century with a few minor modifications and is still used to some extent today, making it the longest-running textbook in history.

Considering Euclid's Postulates

One reason that Euclidean geometry was at the center of philosophy, math and science, was its logical structure and its rigor. Thus the details of the logical structure were considered quite important and were subject to close examination.

The first four postulates, or axioms, were very simply stated, but the Fifth Postulate was quite different from the others.

Postulates I-IV

I. A straight <u>line segment</u> can be drawn joining any two points.

II. Any straight <u>line segment</u> can be extended indefinitely in a straight <u>line</u>.

III. Given any straight <u>line segment</u>, a <u>circle</u> can be drawn having the segment as <u>radius</u> and one endpoint as center.

IV. All <u>right angles</u> are congruent.

Postulate V

V.   If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

A picture of Postulate V

If the sum of angles 1 and 2 is less than a straight angle, then the lines intersect at a point C on the same side of the line AB as the two angles.

Why Postulate V is the Parallel Postulate

- The Postulate does not mention the word parallel, but for a line m through A and any line n through a point B not on m, this rules out the possibility that line n is parallel to m except when two interior angles add up to a straight angle.

- So there is only one possible line through B parallel to m. It can be proved that this line is in fact parallel.

Can One Prove Postulate V from I–IV?

- Postulate V is about 4 times as long as the average length of the first four postulates.

- In fact, its converse is a theorem.

- Many mathematicians and philosophers from Greek times onward felt that this was so complicated that there must be a way to prove Postulate V from the others. The lack of such a proof was a blemish on Euclid.

Attempts to Prove V

The attempts to prove postulate V dig deep into subtleties of basic geometry. Here are some theorems, that if proved, imply V:

- The distance between two parallel lines is finite. (Proclus, 410-485)

- A quadrilateral with two equal sides perpendicular to the base is a rectangle. Saccheri, 1667-1733)

- A quadrilateral with 3 right angles is a rectangle. (Lambert, 1728-1777)

Assuming V False to Prove Euclid Right

- Such mathematicians as Saccheri attempted proofs of Postulate V by contradiction. They assumed that V is false, then proved many theorems based on this assumption -- with the goal of finding a contradiction.

- Saccheri never really found a contradiction but he concluded that Euclid was vindicated anyway because the theorems were so odd.

Assuming V False Leads to a New Geometry

In the 1800s, several mathematicians took a different tack. They assumed Postulate V is false, but then concluded that the resulting theorems were about a new, non-Euclidean, geometry.

Karl Friedrich Gauss, Janos Bolyai, and Nikolai Ivanovich Lobachevsky made this decisive step independently, about the same tme.

Controversy

- The idea of such a new geometry shook mathematics and science to its foundations.

- There was much doubt and debate about this new geometry. It seemed very strange, and there was no proof that the next week a contradiction would not be discovered.

- However, the doubts were resolved by the discovery of models for the new geometry.

Models

A mathematical model of an abstract system such as non-Euclidean geometry is a set of objects and relations that satisfy as theorems the axioms of the system.

Then the abstract system is as consistent as the objects from which the model made. So if a model of non-Euclidean geometry is made from Euclidean objects, then non-Euclidean geometry is as consistent as Euclidean geometry.

Disk Models of non-Euclidean Geometry

Beltrami and Klein made a model of non-Euclidean geometry in a disk, with chords being the lines. But angles are measured in a complicated way.

Poincaré discovered a model made from points in a disk and arcs of circles orthogonal to the boundary of the disk. Angles are measured in the usual way.

Strange Theorems

Thus we can prove theorems in non-Euclidean geometry by proofs about the models.

- Some parallel line pairs have just one common perpendicular and grow far apart. Other parallels get close together in one direction.

- Angle sums of triangles are less than 180 degrees. There are no rectangles at all.

Curvature

Riemann was the first geometer who really sorted out a key concept in geometry. He made a general study of curvature of spaces in all dimensions. In 2-dimensions:

- Euclidean geometry is flat (curvature = 0) and any triangle angle sum = 180 degrees.

- The non-Euclidean geometry of Lobachevsky is negatively curved, and any triangle angle sum < 180 degrees.

- The geometry of the sphere is positively curved, and any triangle angle sum > 180 degrees.

Hyperbolic?

- The negatively curved non-Euclidean geometry is called hyperbolic geometry.

- Euclidean geometry in this classification is parabolic geometry, though the name is less-often used.

- Spherical geometry is called elliptic geometry, but the space of elliptic geometry is really has points = antipodal pairs on the sphere. With this idea, two lines really intersect in a point.

Why is non-Euclidean Geometry Important?

- The discovery of non-Euclidean geometry opened up geometry dramatically. These new mathematical ideas were the basis for such concepts as the general relativity of a century ago and the string theory of today.

- The idea of curvature is a key mathematical idea. Plane hyperbolic geometry is the simplest example of a negatively curved space. Spherical geometry has even more practical applications.