Math 124A **Midterm Exam** July 19, 2007

Your Name ______________________________

Student Number ______________________________

1. Print your name and student ID number on this page. Do **NOT** separate the pages of the exam.
2. **SHOW ALL OF YOUR WORK.** Partial credit will only be given where you have made it clear that you understand part of the solution. Answers without justification may not receive full credit. **Place a box around your final answer to each equation.**
3. If you use a trial and error (or guess and check) method when an algebraic method is available, you will not receive full credit.
4. You may use one sheet (two-sided) of handwritten notes and a scientific, nongraphing calculator. Do not share notes. Books and other notes are not allowed. If you need more space to solve a problem, use the back of the page preceding that problem.
5. Read each question carefully. Work the problems in an order that will maximize your score. Good Luck!

Score

1.	(10)	
2.	(10)	
3.	(10)	
4.	(10)	
Total	(40)	

1. (10 points) Compute the derivative of each of the following functions by using the differentiation rules we have learned in class.

(a) $f(x) = x^2 \sin(x) + 4e^x$
By the product rule,

$$f'(x) = 2x\sin(x) + x^2\cos(x) + 4e^x.$$

(b) $f(x) = \frac{\tan(x)}{3x^3+5x}$
By the quotient rule,

$$f'(x) = \frac{(3x^3+5x)(\sec^2(x)) - (\tan(x))(9x^2+5)}{(3x^3+5x)^2}.$$

(c) $f(x) = \frac{\sin(x)}{\csc(x)} + \frac{\cos(x)}{\sec(x)}$
Observe that $\frac{\sin(x)}{\csc(x)} = \sin^2(x)$ and $\frac{\cos(x)}{\sec(x)} = \cos^2(x)$. Then $f(x) = \sin^2(x) + \cos^2(x) = 1$ so $f'(x) = 0$.

2. (10 points) Using the fact that

$$\lim_{h\to 0}\frac{2^h-1}{h}=\ln(2),$$

find the equation of the tangent line to the curve $y=2^x$ at the point $x=3$.

Using the limit definition of the derivative to find the slope of the tangent line to the curve $f(x)=2^x$ at the point $x=3$, we get

$$\begin{aligned}
f'(3) &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h}\\
&= \lim_{h\to 0}\frac{2^{3+h}-2^3}{h}\\
&= \lim_{h\to 0}\frac{2^3 2^h-2^3}{h}\\
&= \lim_{h\to 0}\frac{2^3(2^h-1)}{h}\\
&= 2^3\cdot\lim_{h\to 0}\frac{2^h-1}{h}\\
&= 2^3\cdot\ln(2)\\
&= 8\ln(2).
\end{aligned}$$

We know the tangent line goes through the point $(3, f(3))$ so the equation of the tangent line is

$$y-8=8\ln(2)(x-3).$$

3. (10 points) Consider the function

$$f(x) = \begin{cases} \ln(x+1)\sin(\frac{1}{x}) + e^x\cos(x) & \text{for } x > 0 \\ c \cdot (x+3)^2 - \frac{\sin(x)}{x} & \text{for } x < 0 \\ d & \text{for } x = 0. \end{cases}$$

Are there values for the **constants** c and d that will make $f(x)$ continuous at $x = 0$?

In order for f to be continuous at $x = 0$, we need to find values for c and d such that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0).$$

To compute $\lim_{x\to 0^+} \ln(x+1)\sin(\frac{1}{x})$, we need to use the squeeze theorem. For all $x \neq 0$, we know that $-1 \leq \sin(\frac{1}{x}) \leq 1$ so

$$-\ln(x+1) \leq \ln(x+1)\sin(\frac{1}{x}) \leq \ln(x+1).$$

Since the function $\ln(x+1)$ is continuous at $x = 0$, $\lim_{x\to 0^+} \ln(x+1) = \ln(1) = 0$. Therefore, by the squeeze theorem, $\lim_{x\to 0^+} \ln(x+1)\sin(\frac{1}{x}) = 0$. Thus

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \ln(x+1)\sin(\frac{1}{x}) + \lim_{x\to 0^+} e^x\cos(x) \\ &= 0 + \lim_{x\to 0^+} e^x\cos(x) \\ &= e^0\cos(0) = 1. \end{aligned}$$

Now we compute the left-hand limit of $f(x)$:

$$\begin{aligned} \lim_{x\to 0^-} c\cdot(x+3)^2 - \frac{\sin(x)}{x} &= 9c - \lim_{x\to 0^-} \frac{\sin(x)}{x} \\ &= 9c - 1. \end{aligned}$$

Now in order for $f(x)$ to be continuous, we need $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$, meaning $9c - 1 = 1$. Thus

$$c = \frac{2}{9}.$$

In addition, we need $f(0) = \lim_{x\to 0^+} f(x)$ so

$$d = 1.$$

4. (10 points) Steve drops an apple from the Space Needle. The apple's height above the ground (in meters) t seconds after Steve drops it is given by the equation

$$y = -5t^2 + 35t + 150.$$

(a) How long does it take for the apple to reach the ground?
The apple hits the ground when $y = 0$. So we solve for t when

$$-5t^2 + 35t + 150 = 0.$$

Factoring the quadratic gives

$$5(t + 3)(-t + 10) = 0$$

so $t = -3$ or $t = 10$. The solution $t = -3$ doesn't make sense, so it takes 10 seconds for the apple to hit the ground.

(b) What is the apple's average velocity during its fall?
Average velocity is the ratio of change in distance to change in time. Since the apple falls for 10 seconds, its average velocity is

$$v = \frac{y(10) - y(0)}{10 - 0} = \frac{0 - 150}{10} = -15m/s.$$

(c) What is the apple's instantaneous velocity just before it hits the ground?
To find instantaneous velocity, we differentiate the position function with respect to time:

$$y' = -10t + 35.$$

The apple's instantaneous velocity just as it hits the ground is given by $y'(10) = -65m/s$.