Math 324A Spring 2008
Final Exam
June 11, 2008

Your Name ____________________

ID Number ____________________

1. Print your name and student ID number on this page. Do **NOT** separate the pages of the exam.
2. **SHOW ALL OF YOUR WORK.** Partial credit will only be given where you have made it clear that you understand part of the solution. Answers without justification may not receive full credit. **Place a box around your final answer to each equation.**
3. If you use a trial and error (or guess and check) method when an algebraic method is available, you will not receive full credit.
4. You may use one sheet (two-sided) of handwritten notes and a basic scientific calculator. **Do not share notes. Graphing calculators are not allowed.** If you need more room solve a problem, use the back of the page preceding that problem.
5. Leave all of your answers in exact form. For example, sin(1) is exact; but 0.841 is not exact.
6. Read each question carefully. Work the problems in an order that will maximize your score. Good Luck!

Score

1.	(10)	
2.	(10)	
3.	(10)	
4.	(10)	
5.	(10)	
6.	(10)	
Total	(60)	

1. (10 points) Consider the vector field

$$\overrightarrow{F}(x,y,z) = \langle y\cos(xy), x\cos(xy), -\sin(z)\rangle.$$

(a) Show that $\overrightarrow{F}$ is conservative. The curl of F is

$$\begin{aligned}\text{curl}F &= \langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\rangle \\ &= \langle 0, 0, (\cos(xy) - xy\sin(xy)) - (\cos(xy) - xy\sin(xy))\rangle \\ &= \langle 0,0,0\rangle.\end{aligned}$$

Since $\text{curl}F = \overrightarrow{0}$ and F is defined on all of $\mathbb{R}^3$, we know that F must be conservative.

(b) Find a function $f(x,y,z)$ for which $\overrightarrow{F} = \nabla f$.
Integrate the first component function $P = y\cos(xy)$ with respect to x to get that

$$f(x,y,z) = \sin(xy) + g(y,z).$$

Differentiating this with respect to y gives

$$Q = \frac{\partial f}{\partial y} = x\cos(xy) + \frac{\partial g}{\partial y}.$$

Differentiating with respect to z gives

$$R = \frac{\partial f}{\partial z} = 0 + \frac{\partial g}{\partial z}.$$

Thus $\frac{\partial g}{\partial z} = -\sin(z)$ and $g = \cos(z)$ and

$$f(x,y,z) = \sin(xy) + \cos(z).$$

(c) Let C be any curve from the point $(1, \frac{\pi}{4}, 0)$ to the point $(3, \frac{\pi}{6}, \pi)$. What is $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$?

By the fundamental theorem of line integrals,

$$\int_C F \cdot dr = f(3, \frac{\pi}{6}, \pi) - f(1, \frac{\pi}{4}, 0) = -\frac{\sqrt{2}}{2} - 1.$$

2. (10 points) Let C be the triangular path obtained by walking from the point $(0,2)$ to the point $(2,0)$ to the point $(0,-2)$ and then back to $(0,2)$. Evaluate

$$\int_C (2y - e^{-x})dx + (x^2 + \sin^2 y)dy.$$

SOLUTION:

Notice that C traverses the triangular region R bounded by the lines $x = 0$, $y = 2 - x$, and $y = x - 2$ in the clockwise direction. By Green's Theorem, with $P = 2y - e^{-x}$ and $Q = x^2 + \sin^2 y$, we get

$$\begin{aligned}
\int_C Pdx + Qdy &= -\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA \\
&= -\iint_R 2x - 2dA \\
&= -\int_{x=0}^{2} \int_{y=x-2}^{2-x} 2x - 2dydx \\
&= \ldots \\
&= \frac{8}{3}.
\end{aligned}$$

Here we add the minus sign in the first line because C traverses R with a negative orientation.

3. (10 points) Let $\overrightarrow{F}(x,y,z) = \langle x^3 + \sin(yz), y^3 + \cos(xz), z^3 + e^{2xy} \rangle$, and let S be the sphere $x^2 + y^2 + z^2 = 25$. Evaluate $\iint_S \overrightarrow{F} \cdot dS$. (Hint: Don't try to evaluate the integral directly.)

SOLUTION:

Let E be the solid ball $x^2 + y^2 + z^2 \leq 25$, which has boundary surface S. By the divergence theorem,

$$\iint_S \overrightarrow{F} \cdot dS = \iiint_E \text{div} F dV.$$

We compute

$$\text{div} F = \nabla \cdot F = 3x^2 + 3y^2 + 3z^2.$$

Now we use the Divergence Theorem and convert to spherical coordinates:

$$\begin{aligned}
\iint_S \overrightarrow{F} \cdot dS &= \iiint_E \text{div} F dV \\
&= \iiint_E 3(x^2 + y^2 + z^2) dV \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^5 3\rho^2 \rho^2 \sin \phi d\rho d\phi d\theta \\
&= \cdots \\
&= 7500\pi.
\end{aligned}$$

4. (10 points) An ant stands at the center of a circular plate that is spinning at a rate of π radians per second. A bread crumb sits on the edge of the plate. The ant walks towards the bread crumb along a radial line at a rate of 2 centimeters per second relative to the plate. Assume that at the instant when the ant starts walking, the bread crumb is positioned on the positive x-axis. As the ant walks, its speed relative to the ground is given by

$$s = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

What is the ant's speed when it is one centimeter from the center of the plate? Hint: Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ separately.

SOLUTION:

For this problem, the goal is to use the Chain Rule for polar coordinates.

Converting to polar coordinates, we get $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The chain rule gives us

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \theta}\frac{d\theta}{dt} \\ &= \cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt} \end{aligned}$$

and

$$\begin{aligned} \frac{dy}{dt} &= \frac{\partial y}{\partial r}\frac{dr}{dt} + \frac{\partial y}{\partial \theta}\frac{d\theta}{dt} \\ &= \sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt}. \end{aligned}$$

In the problem, we are told that $\frac{d\theta}{dt} = \pi$ radians per second and $\frac{dr}{dt} = 2$ centimeters per second. When the ant is one centimeter from the center of the plate, he has walked one centimeter relative to the plate, and so .5 seconds have elapsed since he started walking. Since the plate spins at π radians per second, this means that he is at the point with polar coordinates $r = 1$ (since he is 1cm from the center of the plate) and $\theta = \frac{\pi}{2}$.

Thus the above equations tell us that $\frac{dx}{dt} = -\pi$ and $\frac{dy}{dt} = 2$ so

$$s = \sqrt{\pi^2 + 4}.$$

5. (10 points) A hiker stands on the side of a mountain whose shape is given by the equation $z = 1500 - .1x^2 - .08y^2$ with x, y and z measured in meters. Assume that the positive x-axis points east and that the positive y-axis points north.

(a) If the hiker walks due north from the point $(5, -5)$, at what rate does the height of the mountain change?

Compute $\nabla f = \langle -.2x, -.16y \rangle$ and $\nabla f(5, -5) = \langle -1, .8 \rangle$. When the hiker walks due north, he heads in the direction of the unit vector $\langle 0, 1 \rangle$ and the rate at which the height of the mountain changes is

$$D_{\langle 0,1 \rangle} f(5, -5) = \langle -1, .8 \rangle \cdot \langle 0, 1 \rangle = .8$$

(b) In what direction from the point $(5, -5)$ does the height of the mountain increase most rapidly?

The height of the moutain increases most rapidly in the direction of $\nabla f(5, -5) = \langle -1, .8 \rangle$.

(c) At what rate will the height of the mountain change if the hiker climbs in the direction you found in part (b)?

The rate at which the height changes is $|\nabla f(5, -5)| = \sqrt{1^2 + .8^2} = \sqrt{1.64}$.

6. (10 points) Compute $\iint_S \text{curl}\overrightarrow{F} \cdot d\mathbf{S}$ where S is the portion of the cone $z = \sqrt{x^2+y^2}$ with $z \geq 0$ below the sphere $x^2+y^2+z^2 = 2$ and $\overrightarrow{F}(x,y,z) = \langle x^2+y^2, ze^{x^2+y^2}, e^{x^2+z^2}\rangle$. Suppose S is oriented so that its normal vectors are pointing downwards.

SOLUTION:

We want to use Stokes' Theorem for this one. The boundary curve of S is the curve where the cone $z = \sqrt{x^2+y^2}$ intersects the sphere $x^2+y^2+z^2 = 2$, we get

$$2 = x^2+y^2+z^2 = z^2+z^2,$$

and hence $z = 1$. Restricting to $z = 1$ says that the cone intersects the sphere in the curve C where $x^2+y^2 = 1$ and $z = 1$, which is a circle.

We parameterize this intersection by $r(t) = \langle \cos t, \sin t, 1\rangle$ so that $r'(t) = \langle -\sin t, \cos t, 0\rangle$. A quick check using the right hand rule tells us that since the normal vectors to S point downwards, the induced orientation on C runs in the clockwise direction. Our parameterization of C runs in the counterclockwise direction, so Stokes Theorem tells us

$$\begin{aligned}
\iint_S \text{curl}\overrightarrow{F} \cdot dS &= \int_C \overrightarrow{F} \cdot dr \\
&= -\int_0^{2\pi} F(r(t)) \cdot r'(t) dt \\
&= -\int_0^{2\pi} \langle 1, e^1, e^{\cos^2 t+1}\rangle \cdot \langle -\sin t, \cos t, 0\rangle dt \\
&= -\int_0^{2\pi} -\sin t + e\cos t dt \\
&= ... \\
&= 0.
\end{aligned}$$