KODAIRA DIMENSION OF SUBVARIETIES

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1. Introduction

One goal in the study of algebraic geometry is to classify varieties up to isomorphism. Since every singular variety has a non-singular resolution and every quasi-projective variety lies in a complete variety, a way to start this project is to classify non-singular complete varieties.

In the case of curves this has been done. In the set of complete non-singular curves birational equivalence is equivalent to isomorphism. In this set, curves are classified first by a discrete invariant, their genus, $g \in \mathbb{Z}$, $g \geq 0$, and then by a connected moduli space, $\mathscr{M}_g$.

The case of surfaces is more difficult. It can first be shown that every birational equivalence class of surfaces not containing a ruled surface has within it a unique 'minimal model'. Then it is necessary to classify birationally ruled surfaces and to classify non-birationally ruled surfaces by their minimal models.

Similarly in higher dimensions there is a strategy of finding 'minimal models' within birational classes and then classifying the birational classes of varieties of a given dimension.

In classifying birational classes what seems to be the first distinction to make after dimension is a birational invariant called the Kodaira dimension of a variety. We will start this paper by discussing this invariant.

2. D-Dimension and Kodaira Dimension

Let X be a complete variety over an algebraically closed field k. Let D be an effective Cartier divisor on X with corresponding invertible sheaf $\mathscr{L}(D)$. Since D is effective, this sheaf induces a rational map:

$$\phi_D : X \dashrightarrow \mathbb{P}^N, \text{ where } N = \dim H^0(X, \mathscr{L}(D)) - 1$$

This map is modelled by $U \to \mathbb{P}^N$, where $X \setminus U \subseteq X$ is the closed set where all the global sections of $\mathscr{L}(D)$ vanish.

For any divisor D on X we may consider for which $m \in \mathbb{N}$ the divisor mD is effective, and then consider the rational maps ϕ_{mD}. This is done in the following definitions.

Definition 2.1. *For any $\mathbb{Q}$-Cartier divisor D on X, let:*

$$\mathbb{N}(D) = \{m \in \mathbb{N} \mid mD \textit{ is effective and Cartier}\}$$

It is clear that $\mathbb{N}(D)$ is a semigroup. If $\mathbb{N}(D) \neq \emptyset$, let $m_0 = \gcd\{m \in \mathbb{N}(D)\}$. Then it is easy to show that there exists $m' > 0$ such that $mm_0 \in \mathbb{N}(D)$ for $m \geq m'$.

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Definition 2.2. *Let* $\kappa(D,X) = \max\{\dim \operatorname{im} \phi_{mD} \mid m \in \mathbb{N}(D)\}$*. We say* $\kappa(D,X) = -\infty$ *if* $\mathbb{N}(D) = \emptyset$*. This is called the* D*-dimension of* X*.*

The Kodaira dimension invariant we are interested in is a special case of D-dimension as given in the following definition.

Definition 2.3. *The Kodaira dimension of a complete variety* X *is defined as* $\kappa(X) = \kappa(K_X, X)$*, where* K_X *is a cannonical divisor on* X*.*

We will show in the next section that the Kodaira dimension of a variety is actually a birational invariant.

3. Asymptotic Relation for $\kappa(D,X)$ and $\kappa(X)$

The D-dimension, $\kappa(D,X)$ can also be expressed in terms of the rate of growth of $h^0(X, \mathscr{L}(mm_0D)) = \dim H^0(X, \mathscr{L}(mm_0D))$ as m increases. This is shown in the following theorem:

Theorem 3.1. *Let* X *be a complete variety and* D *a divisor on* X *with* $\kappa = \kappa(D,X) \geq 0$*, then there exists* $\alpha, \beta > 0$ *such that*

$$\alpha m^\kappa < h^0(X, \mathscr{L}(mm_0D)) < \beta m^\kappa$$

for $m >> 0$*.*

Proof. We will prove this theorem in the case that $\mathscr{L}(m_1D)$ is generated by global sections on X for some $m_1 = am_0 > 0$. For the case in which $|mD|$ has a fixed component for all $m \in \mathbb{N}(D)$ a proof can be found in [3, §10.2].

Thus assume that $\mathscr{L}(m_1D)$ is generated by global sections. Then $\phi_{m_1D} : X \dashrightarrow \mathbb{P}^N$ is a morphism. If H is a hyperplane divisor of $\mathbb{P}^N$, then $\phi_{m'm_1D} = \phi_{m'H} \circ \phi_{m_1D}$, where $\phi_{m'H}$ is the m'-uple embedding.

This means:

$$\dim \operatorname{im} \phi_{m_1D} = \dim \operatorname{im} \phi_{m'm_1D} = \kappa(D,X) = \kappa$$

since for any $m' > 0$,

$$\dim \operatorname{im} \phi_{m'D} = \kappa(D,X) \implies \dim \operatorname{im} \phi_{m'm_1D} = \kappa(D,X)$$

In this case we also have $\mathscr{L}(m'm_1D) = \phi^*_{m_1D}(\mathscr{O}_{\mathbb{P}^N}(m'))$. This means for $m' >> 0$:

$$\begin{aligned} h^0(X, \mathscr{L}(m'm_1D)) &= h^0(X, \phi^*_{m_1D}(\mathscr{O}(m'))) \\ &= h^0(\phi_{m_1D}(X), \mathscr{O}(m')) \\ &= P_{\phi_{m_1D}(X)}(m'), \end{aligned}$$

where $P_{\phi_{m_1D}(X)}$ is the Hilbert polynomial of $\phi_{m_1D}(X) \subseteq \mathbb{P}^N$.

Since $\dim \phi_{m_1D}(X) = \kappa$, the polynomial $P_{\phi_{m_1D}(X)}$ has degree κ. Thus we may choose $\alpha', \beta' > 0$ such that:

$$\alpha' m'^\kappa < h^0(X, \mathscr{L}(m'm_1D)) < \beta' m'^\kappa,$$

for $m' >> 0$.

Given a large m, we may write $m = m'a + r$ with $0 \leq r < a$. In this case $mm_0D = m'm_1D + rm_0D$, and we have:

$$\alpha'\left(\frac{1}{2a}\right)^\kappa m^\kappa < \alpha' m'^\kappa < h^0(X, \mathscr{L}(m'm_1D))$$

$$\leq h^0(X, \mathscr{L}(mm_0D))$$
$$\leq h^0(X, \mathscr{L}((m'+1)m_1D)) < \beta'(m'+1)^\kappa < \beta'\left(\frac{2}{a}\right)^\kappa m^\kappa$$

Thus if we choose $\alpha = \alpha'\left(\frac{1}{2a}\right)^\kappa$, and $\beta = \beta'\left(\frac{2}{a}\right)^\kappa$, the theorem is proven in our special case. □

We see then that if $\kappa(D, X) \geq 0$, it is the same as the degree of the Hilbert polynomial of $R(D, X)^{(m_0)} = R(m_0D, X)$, where

$$R(D, X) = \bigoplus_{m=0}^{\infty} H^0(X, \mathscr{L}(mD))$$

is a graded ring over k.

Since the degree of the Hilbert polynomial of $R(D, X)^{(m_0)}$ is $\kappa(D, X)$, we see by commutative algebra that tr. $\deg_k R(D, X)^{(m_0)} =$ tr. $\deg_k R(D, X) = \kappa(D, X) + 1$ In the case that $D = K_X$, we call

$$R(K_X, X) = \bigoplus_{m=0}^{\infty} H^0(X, \omega_X^{\otimes m})$$

the plurigenus ring of X

Given a rational map, $X \dashrightarrow X'$, with model $f : U \to X'$, we may consider the induced map of sheaves $f^*\omega_{X'} \to \omega_U$. Doing this leads to an argument that $h^0(X, \omega_X^{\otimes m})$ is a birational invariant. This argument for $h^0(X, \omega_X)$ can be found in [2, II.8.19].

Along with the observation that $\kappa(X) = \kappa(K_X, X) =$ tr. $\deg_k R(K_X, X) - 1$, if $R(K_X, X) \neq R(K_X, X)_0$, and $\kappa(X) = -\infty$ otherwise, this proves the following theorem

Theorem 3.2. *The Kodaira dimension, $\kappa(X)$, of a complete variety X is a birational invariant.*

4. Easy Addition Theorem

Next we will look at an important theorem concerning Kodaira dimension. The Easy Additition Theorem gives a relation between the Kodaira dimension of the general fibers of certain surjective maps and that of the domain varieties. Before stating and proving a version of this theorem we will give an alternate definition of D-dimension. What follows in this section is adapted from the presentations in [4, §1] and [3, §10.3].

For any irreducible reduced scheme X over an algebraically closed k and Cartier divisor D we may define the graded ring $R(D, X)$ as above. Let $Q(D, X)$ be the quotient ring of $R(D, X)$. Let $S \subseteq R(D, X)$ be the multiplicative subset of non-zero homogeneous elements, and let $Q((D, X)) = (S^{-1}R(D, X))_0$ be the field of degree 0 elements in the quotient ring $S^{-1}R(D, X)$.

If $R(D, X)$ is not trivial (all in degree 0), $Q(D, X)$ will be a field extension of $Q((D, X))$ in one trancendental variable. So we have: tr. $\deg_k Q((D, X)) =$ tr. $\deg_k Q(D, X) - 1$

Thus we may give the following alternate definition of $\kappa(D, X)$:

$$\kappa(D, X) = \begin{cases} -\infty & \text{if } R(D, X) = R(D, X)_0 \\ \text{tr. } \deg_k Q((D, X)) & \text{otherwise} \end{cases}$$

We will use this definition in the proof of the Easy Addition Theorem

Theorem 4.1 (Easy Addition)**.** *Let $f : X \to Y$ be a dominant proper morphism of non-singular varieties over an algebraically closed field, k, of characteristic 0, and let D be a Cartier divisor on X, then:*

$$\kappa(D, X) \leq \kappa(D_y, X_y) + \dim Y \quad \textit{for all } y \in Y',$$

where $f' : X' \to Y'$ is the smooth part of f.

Proof. If $\kappa(D, X) = -\infty$, the theorem is clear. Thus we will assume $\kappa(D, X) \geq 0$.

In this case, let η be the generic point of Y, and let X_η be the pull-back, so we have a diagram:

$$\begin{array}{ccccc} X_\eta & \longrightarrow & X' & \longrightarrow & X \\ \downarrow^f & & \downarrow^{f'} & & \downarrow^f \\ \operatorname{Spec} K(Y) & \longrightarrow & Y' & \longrightarrow & Y \end{array}$$

Let $\mathscr{L}$ be any invertible sheaf on X. Then $f_*\mathscr{L}|_{Y'}$ will be flat and coherent over Y'. This means the restriction map $r : H^0(Y', f_*\mathscr{L}|_{Y'}) \to f_*\mathscr{L} \otimes K(Y)$ is injective. Furthermore, since $\mathscr{L}$ is invertible, the restriction $H^0(X, \mathscr{L}) \to H^0(X', \mathscr{L}|_{X'})$ is also injective. We have:

$$H^0(X, \mathscr{L}) \hookrightarrow H^0(X', \mathscr{L}|_{X'}) \simeq H^0(Y', f_*\mathscr{L}|_{Y'}) \hookrightarrow f_*\mathscr{L} \otimes K(Y) \simeq H^0(X_\eta, \mathscr{L}_\eta)$$

Applying this for $\mathscr{L} = \mathscr{L}(mD)$ implies that $R(D, X) \subseteq R(D_\eta, X_\eta)$ and $Q((D, X)) \subseteq Q((D_\eta, X_\eta))$. In particular, $\text{tr. deg}_k Q((D, X)) \leq \text{tr. deg}_k Q((D_\eta, X_\eta))$ If we consider X_η as a variety over $K(Y)$, we have:

$$\kappa(D, X) = \text{tr. deg}_k Q((D, X)) \leq \text{tr. deg}_k Q((D_\eta, X_\eta))$$
$$= \text{tr. deg}_{K(Y)} Q((D_\eta, X_\eta)) + \text{tr. deg}_k K(Y) = \kappa(D_\eta, X_\eta) + \dim Y$$

Since $f' : X' \to Y'$ is flat, we have, as shown in [2, III128], for an invertible sheaf, $\mathscr{L}$, on X: $h^0(X_y, \mathscr{L}_y) = \dim_{k(y)} H^0(X_y, \mathscr{L}_y)$ is an upper semicontinuous function on Y.

In particular for any $y \in Y',\ m \in \mathbb{N}$:

$$h^0(X_\eta, \mathscr{L}(mD_\eta)) \leq h^0(X_y, \mathscr{L}(mD_y))$$

This means the Hilbert polynomial of $R(D_y, X_y)$ grows at least as fast as that of $R(D_\eta, X_\eta)$. We have:

$$\kappa(D_\eta, X_\eta) \leq \kappa(D_y, X_y) \quad \Longrightarrow \quad \kappa(D, X) \leq \kappa(D_y, X_y) + \dim Y$$

□

5. Results of Peternell, Schneider and Sommese

Let $f : X \to Y$ be a dominant proper map as is theorem 4.1, and let $F \subseteq X$ be a general fiber. Then since $K_X|_F \sim K_F$ [3, §10.5], the Easy Addition Theorem gives us:

$$\kappa(X) \leq \kappa(F) + \text{codim}_{F|X}$$

In a 1999 paper [5], Peternell, Schneider and Sommese gave a generalization of this result for $Y \subseteq X$, where Y is not necessarily the fiber of a map. A less than fully general version of their result can be stated as:

Theorem 5.1 (Peternell, Schneider, Sommese)**.** *Let X be a non-singular projective variety and $Y \subseteq X$ a non-singular subvariety. Then if $\mathscr{N}_{Y|X}$ is $\mathbb{Q}$-effective, we have $\kappa(X) \leq \kappa(Y) + \text{codim}_{Y|X}$.*

Here we use the following definition:

Definition 5.2. *If X is a non-singular complete variety and $\mathscr{F}$ is a locally free coherent sheaf on X, then we say $\mathscr{F}$ is $\mathbb{Q}$-effective if for some $k_0 \in \mathbb{N}$, $\operatorname{Sym}^{k_0} \mathscr{F}$ is generically spanned, i.e. $H^0(X, \operatorname{Sym}^{k_0} \mathscr{F}) \to \operatorname{Sym}^{k_0} \mathscr{F} \otimes K(X)$ is surjective where $\operatorname{Sym}^{k_0} \mathscr{F}$ is the k_0th symmetric tensor power of $\mathscr{F}$ and η is the generic point of X.*

To prove the theorem we will proceed in two steps, each given in a proposition.

Proposition 5.3. *Let $Y \subseteq X$ as in the theorem, and let $\mathscr{L}$ be an invertible sheaf on X. Then there exists $c \in \mathbb{N}$ such that for all $t \geq 0$:*

$$h^0(X, \mathscr{L}^t) \leq \sum_{k=0}^{ct} h^0(Y, \mathscr{L}^t|_Y \otimes \operatorname{Sym}^k \mathscr{N}^*_{Y|X})$$

Let $\mathscr{I}_Y \subseteq \mathscr{O}_X$ be the ideal sheaf of $Y \subseteq X$. Then we have a short exact sequence of sheaves:

$$0 \to \mathscr{I}_Y \to \mathscr{O}_X \to \mathscr{O}_Y \to 0$$

Let $t \geq 0$ and $k \in \mathbb{N}$. Tensoring the above sequence by $\mathscr{L}^t \otimes \mathscr{I}_Y^k$ gives:

$$0 \to \mathscr{L}^t \otimes \mathscr{I}_Y^{k+1} \to \mathscr{L}^t \otimes \mathscr{I}_Y^k \to \mathscr{L}^t|_Y \otimes \operatorname{Sym}^k \mathscr{N}^*_{Y|X}$$

When $k = 0$, the cohomology sequence gives us:

$$h^0(X, \mathscr{L}^t) \leq h^0(Y, \mathscr{L}^t|_Y) + h^0(X, \mathscr{L}^t \otimes \mathscr{I}_Y)$$

Proceeding by induction on $n \in \mathbb{N}$ where this is the base case, we conclude:

$$h^0(X, \mathscr{L}^t) \leq \sum_{k=0}^{n} h^0(Y, \mathscr{L}^t|_Y \otimes \operatorname{Sym}^k \mathscr{N}^*_{Y|X}) + h^0(X, \mathscr{L}^t \otimes \mathscr{I}_Y^{n+1}) \quad \text{for all } n \in \mathbb{N}$$

Thus we have proved the proposition if we prove the following lemma:

Lemma 5.4. *There exists $c \in \mathbb{N}$ such that for all $t \geq 0$ and $k > ct$:*

$$h^0(X, \mathscr{L}^t \otimes \mathscr{I}_Y^k) = 0$$

To prove this lemma we will need still another lemma and a definition.

Definition 5.5. *A curve C in X is called a very ample curve if it is the intersection of $\dim X - 1$ very ample divisors.*

Lemma 5.6. *If $C \subseteq X$ is a very ample curve, and $\mathscr{F}$ is a coherent sheaf. Then:*

$$H^0(C, \mathscr{F}|_C) = 0 \implies H^0(X, \mathscr{F}) = 0$$

Proof. Since C is the intersection of $\dim X - 1 = r$ divisors on X, we may write $C = H_r \subseteq H_{r-1} \subseteq \ldots \subseteq H_1 \subseteq H_0 = X$, where each H_i is an ample divisor on H_{i-1}. Now we have a short exact sequence:

$$0 \to \mathscr{L}(-H_1) \otimes \mathscr{F} \to \mathscr{F} \to \mathscr{F}|_{H_1} \to 0$$

Thus if $H^0(H_1, \mathscr{F}|_{H_1}) = 0$, we have $H^0(X, \mathscr{L}(-H_1) \otimes \mathscr{F}) \simeq H^0(X, \mathscr{F})$. Induction shows that:

$$H^0(X, \mathscr{L}(-H_1)^n \otimes \mathscr{F}) \simeq H^0(X, \mathscr{F}) \quad \text{for all } n \in \mathbb{N}$$

But the left side is eventually zero, so we have $H^0(H_1, \mathscr{F}|_{H_1}) = 0 \implies H^0(X, \mathscr{F}) = 0$. Then the lemma follows by induction. ☐

Proof. (of Lemma 5.4) For a first case we will assume Y is a divisor. By Bertini's Theorem, we may choose a very ample curve $C \subseteq X$ such that C is non-singular, connected, and meets Y transversely. In this case we have:

$$\deg(\mathscr{L}^t \otimes \mathscr{I}_Y^k)|_C = t \deg \mathscr{L}|_C - kY.C$$

Since C is a very ample curve, we will have $Y.C > 0$. Then for $k > \left(\frac{\deg \mathscr{L}|_C}{Y.C}\right)t$, we will have:

$$H^0(C, (\mathscr{L}^t \otimes \mathscr{I}_Y^k)|_C) = 0 \implies H^0(X, \mathscr{L}^t \otimes \mathscr{I}_Y^k) = 0$$

Thus we take $c = \left(\frac{\deg \mathscr{L}|_C}{Y.C}\right)$.

If Y is not a divisor, let $\tilde{X}$ be the blow-up of X at Y and Y' be the inverse image of Y. Then Y' is a divisor on $\tilde{X}$ and $\pi^*\mathscr{L}$ is an invertible sheaf on $\tilde{X}$. Let us choose a very ample curve, $C \subseteq \tilde{X}$ as before, non-singular and meeting Y' transversely. Then by what we have already said we may conclude that there exists $c \in \mathbb{N}$ such that for all $k > ct$:

$$H^0(C, ((\pi^*\mathscr{L})^t \otimes \mathscr{I}_{Y'}^k)|_C) = 0$$
$$\implies H^0(\tilde{X}, (\pi^*\mathscr{L})^t \otimes \mathscr{I}_{Y'}^k) = 0$$
$$\implies H^0(X, \mathscr{L}^t \otimes \mathscr{I}_Y^k) = 0$$

This proves lemma 5.4 and proposition 5.3. □

Proposition 5.7. *Let $X \subseteq Y$ be as in theorem 5.1 with $\mathscr{N}_{Y|X}$ $\mathbb{Q}$-effective, and let $\mathscr{L}$ be an invertible sheaf on X. Then:*

$$\kappa(\mathscr{L}, X) \leq \kappa(\mathscr{L}|_Y, Y) + \mathrm{codim}_{Y|X},$$

where $\kappa(\mathscr{L}, X) = \kappa(D, X)$ for $\mathscr{L} = \mathscr{L}(D)$.

Before proving the proposition, we will give a lemma about the dimension of symmetric powers of vector spaces that will be useful.

Lemma 5.8. *Let V be a vector space over k of dimension d. Then for a fixed d we may choose $\alpha_d, \beta_d > 0$ such that:*

$$\alpha_d m^{d-1} \leq \dim \mathrm{Sym}^m V \leq \beta_d m^{d-1} \quad \textit{for all } m \in N$$

Proof. The lemma follows immediately from the observation that:

$$\dim \mathrm{Sym}^m V = \binom{d+m-1}{d-1}$$

□

Proof. (of proposition) Since $\mathscr{N}_{Y|X}$ is $\mathbb{Q}$-effective, there exists $k_0 \in \mathbb{N}$ such that $\mathrm{Sym}^{k_0} \mathscr{N}_{Y|X}$ is generically spanned. In this case, $\mathrm{Sym}^{k_0 m} \mathscr{N}_{Y|X}$ will also be generically spanned for $m \in \mathbb{N}$. This means there is a map $\mathscr{O}_Y^M \to \mathrm{Sym}^{k_0 m} \mathscr{N}_{Y|X}$, which is generically surjective, where $M = \mathrm{rank}\, \mathrm{Sym}^{k_0 m} \mathscr{N}_{Y|X} < \beta_d (k_0 m)^{d-1}$ for $d = \mathrm{rank}\, \mathscr{N}_{Y|X} = \mathrm{codim}_{Y|X}$. Taking the dual of this map shows that there is an injection: $\mathrm{Sym}^{k_0 m} \mathscr{N}_{Y|X}^* \hookrightarrow \mathscr{O}_Y^M$. This allows us to write the following inequalities for $m \in \mathbb{N}$:

$$\begin{aligned} h^0(Y, \mathscr{L}^t|_Y \otimes \operatorname{Sym}^m \mathscr{N}^*_{Y|X}) &\le h^0(Y, \mathscr{L}^{k_0 t}|_Y \otimes \operatorname{Sym}^{k_0 m} \mathscr{N}^*_{Y|X}) \\ &\le h^0(Y, \mathscr{L}^{k_0 t}|_Y \otimes \mathscr{O}_Y^M) \\ &= (\operatorname{rank} \mathscr{O}_Y^M) h^0(Y, \mathscr{L}^{k_0 t}|_Y) \\ &\le \beta_d (k_0 m)^{d-1} \beta_{\mathscr{L}|_Y} (k_0 t)^{\kappa(\mathscr{L}|_Y, Y)} \\ &= C m^{d-1} t^{\kappa(\mathscr{L}|_Y, Y)} \end{aligned}$$

where $C > 0$ is a constant independent of m and t. (Note that this holds even for $\kappa(\mathscr{L}|_Y, Y) = -\infty$ if we take $t^{-\infty}$ to be 0.)

If we now apply proposition 5.3 with a correctly chosen $c \in \mathbb{N}$ and C as above, we may conclude:

$$h^0(X, \mathscr{L}^t) \le \sum_{k=0}^{ct} C k^{d-1} t^{\kappa(\mathscr{L}|_Y, Y)} \le C' t^{\kappa(\mathscr{L}|_Y, Y) + d}$$

for $C' > 0$ another constant independent of t.

But this means:

$$\kappa(\mathscr{L}, X) \le \kappa(\mathscr{L}|_Y, Y) + \operatorname{codim}_{Y|X}$$

□

Proof. (of Theorem 5.1) We know that $\omega_Y \simeq \omega_X|_Y \otimes \det \mathscr{N}_{Y|X}$. Since $\operatorname{Sym}^{k_0} \mathscr{N}_{Y|X}$ is globally spanned, it follows that $(\det \mathscr{N}_{Y|X})^{k_0}$ is effective. This means for $m \in \mathbb{N}$:

$$h^0(Y, (\omega_X|_Y)^{mk_0}) \le h^0(Y, \omega_Y^{mk_0})$$

It then follows from the asymptotic relation for D-dimension and Kodaira dimension that $\kappa(\omega_X|_Y, Y) \le \kappa(Y)$. Combining this with the last proposition proves theorem 5.1. That is, we have:

$$\kappa(X) = \kappa(\omega_X, X) \le \kappa(\omega_X|_Y, Y) + \operatorname{codim}_{Y|X} \le \kappa(Y) + \operatorname{codim}_{Y|X}$$

□

6. A Viewpoint and Questions

One way to view theorem 5.1 is as a partial answer to the following question.

Question 6.1. *Given a non-singular projective variety Y and a rank r locally free sheaf $\mathscr{N}$ on Y, suppose X is also a non-singular projective variety with $Y \subseteq X$ and $\mathscr{N}_{Y|X} \simeq \mathscr{N}$. What can we say about $\kappa(X)$ in terms of $\kappa(Y)$ and $\mathscr{N}$?*

Theorem 5.1 tells us that in the case that $\mathscr{N}$ is $\mathbb{Q}$-effective, we may conclude that $\kappa(X) \le \kappa(Y) + \operatorname{codim}_{Y|X}$.

This raises two questions:

(1) Is this result sharp? That is, may we always choose X so that $\kappa(X) = \kappa(Y) + \operatorname{codim}_{Y|X}$? If not, under what conditions may we choose such an X?
(2) Can any lower bound be given for $\kappa(X)$?

In the remainder of this paper we will give a partial answer to these questions.

First, in regard to a lower bound for $\kappa(X)$, we will observe that the natural projective completion of the vector bundle associated to $\mathscr{N}^*$ is a variety, X, as described with $\kappa(X) = -\infty$. Thus in the greatest generality the second question is answered in the negative.

From there on we will consider only the case where Y is projective, $\mathrm{codim}_{Y|X} = 1$, and $\mathscr{N} \simeq \mathscr{O}_Y(n)$ for $n \in \mathbb{N}$. We will first give a construction of the completion of the line bundle associated to $\mathscr{N}^*$ as a subvariety of $\tilde{\mathbb{P}}(1, \ldots, 1, n)$, a resolution (or blow-up) of weighted projective space.

Then in the case that Y is a complete intersection of general type, we will show that for many such Y the above variety may be deformed to a non-singular variety, $\tilde{X}$, also of general type. This will give the following proposition:

Proposition 6.2. *Let $Y \subseteq \mathbb{P}^N$ be a non-singular complete intersection variety of multi-degree $(d_1, \ldots, d_r)$ with $d_t > 1$ for all t, and $n \in \mathbb{N}$. If $\sum d_t > n + N + 1$ (and thus Y is of general type), and $d_i > 2n$ for some i, or $d_i, d_j > n$ for some $i \neq j$, then Y lies in a non-singular projective variety $\tilde{X}$ with $\mathscr{N}_{Y|\tilde{X}} \simeq \mathscr{O}_Y(n)$, which is also of general type.*

In this case we find that the formula $\kappa(X) \leq \kappa(Y) + \mathrm{codim}_{Y|X}$ is sharp, since we have found $\tilde{X}$ with:

$$\kappa(\tilde{X}) = \dim \tilde{X} = \dim Y + 1 = \kappa(Y) + \mathrm{codim}_{Y|X}$$

7. The Vector Bundle $\mathbf{V}(\mathscr{N}^*)$ and its Completion

In this section we will answer the second sub-question posed in section 6. In particular, given a complete non-singular variety, Y, and a locally free rank r sheaf on Y, $\mathscr{N}$, we will show that there exists a complete non-singular variety X with $Y \subseteq X$, $\mathscr{N}_{Y|X} \simeq \mathscr{N}$ and $\kappa(X) = -\infty$.

This follows immediately from the following proposition.

Proposition 7.1. *Let Y be a complete non-singular variety and $\mathscr{N}$ be a locally free rank r sheaf on Y. Let $\mathbf{V}(\mathscr{N}^*)$ be the geometric vector bundle over Y associated with $\mathscr{N}^*$. Then the zero section of $\mathscr{N}$, $0 \in H^0(Y, \mathscr{N})$, corresponds to a section, $s : Y \to \mathbf{V}(\mathscr{N}^*)$, of the bundle map, $\pi : \mathbf{V}(\mathscr{N}^*) \to Y$. In this case we have: $\mathscr{N}_{s(Y)|\,\mathbf{V}(\mathscr{N}^*)} \simeq \mathscr{N}$.*

Proof. We start by recalling that $\mathbf{V}(\mathscr{N}^*) = \mathbf{Spec}\,\mathrm{Sym}(\mathscr{N}^*)$, where:

$$\mathrm{Sym}(\mathscr{N}^*) = \bigoplus_{m=0}^{\infty} \mathrm{Sym}^m \mathscr{N}^*$$

is the graded symmetric powers ring of $\mathscr{N}^*$.

Locally, for an affine neighborhood, $U \simeq \mathrm{Spec}\, A \subseteq Y$, we have:

$$\pi^{-1}(U) \simeq \mathrm{Spec}\,\mathrm{Sym}(\mathscr{N}^*(U))$$

These neighborhoods are glued together in the natural way to form X. The zero section, $0 \in H^0(Y, \mathscr{N})$, corresponds to the zero map in $\mathrm{Hom}(\mathscr{N}^*, \mathscr{O}_Y)$ since $\mathscr{N}$ is reflexive. This in turn gives us a surjective map $\mathrm{Sym}(\mathscr{N}^*) \to \mathscr{O}_Y$, which corresponds to a morphism $s : Y \to \mathbf{V}(\mathscr{N}^*)$. This is the section referred to in the proposition.

We have $s(U) \subseteq \pi^{-1}(U) \simeq \operatorname{Spec} S$, where $S = \operatorname{Sym}(\mathscr{N}^*(U))$. It is clear that the ideal of $s(U)$ in S is given by:

$$I_{s(U)} = \ker(S \to A) = S_{>0} = \bigoplus_{m=1}^{\infty} \operatorname{Sym}^m \mathscr{N}^*(U)$$

Since S is generated by $S_0 = A$ and S_1, we will have:

$$I^2_{s(U)} = \bigoplus_{m=2}^{\infty} \operatorname{Sym}^m \mathscr{N}^*(U)$$
$$\text{and} \quad I_{s(U)}/I^2_{s(U)} \simeq \mathscr{N}^*(U)$$

Since the neighborhoods $\pi^{-1}(U)$ glue together naturally, we have $\mathscr{I}_{s(Y)}/\mathscr{I}^2_{s(Y)} \simeq \mathscr{N}^*$. Taking the dual gives us the result. □

Notice that $\pi^{-1}(U) = \operatorname{Spec}\operatorname{Sym}(\mathscr{N}^*(U))$ lies as an open set in $\operatorname{Proj}\operatorname{Sym}(\mathscr{N}^*(U) \oplus tA)$ corresponding to $t \neq 0$.

Similarly if we now let $X = \mathbb{P}(\mathscr{N}^* \oplus \mathscr{O}_Y) = \mathbf{Proj}\operatorname{Sym}(\mathscr{N}^* \oplus \mathscr{O}_Y)$, we see that since $\mathscr{O}_Y$ has a global section on Y, corresponding locally to t, $\mathbf{V}(\mathscr{N}^*)$ will lie as an open set in X. Thus $X = \mathbb{P}(\mathscr{N}^* \oplus \mathscr{O}_Y)$ is a non-singular completion of $\mathbf{V}(\mathscr{N}^*)$

Since Y is proper, $s(Y)$ will be disjoint from $X \setminus \mathbf{V}(\mathscr{N}^*)$. This means we will still have $\mathscr{N}_{s(Y)|X} \simeq \mathscr{N}$. We also have a projection map, $\pi : X \to Y$, with fibers isomorphic to $\mathbb{P}^r$. Since $\kappa(\mathbb{P}^r) = -\infty$, the Easy Addition Theorem tells us that $\kappa(X) = -\infty$.

Thus by specifying only Y and $\mathscr{N}$ it is impossible to put a lower bound on $\kappa(X)$.

For the remainder of this paper we will restrict ourselves to the situation in which $Y \subseteq \mathbb{P}^N$ and $\mathscr{N} \simeq \mathscr{O}_Y(n)$ for some $n \in \mathbb{N}$. In particular we will have $\operatorname{codim}_{Y|X} = 1$. First we will give an explicit construction of the completion of $\mathbf{V}(\mathscr{N}^*)$ as described above. To do this we will work in weighted projective space.

First consider a non-singular projective variety, $Y = \operatorname{Proj} k[x_0, \ldots, x_N]/I_Y$, with associated ideal I_Y, and let $\mathscr{N} \simeq \mathscr{O}_Y(n)$ for $n \in \mathbb{N}$.

$$\text{Let} \quad \mathbb{P} = \mathbb{P}(1, \ldots, 1, n) = \operatorname{Proj} k[x_0, \ldots, x_N, s] \quad (\deg x_i = 1,\ \deg s = n)$$

This is a weighted projective space. Notice that if $n > 1$, $\mathbb{P}$ has a singularity at $Q = (0 : \ldots : 0 : 1)$. Let $I_{C(Y)} = k[x_0, \ldots, x_N, s]I_Y$ be the ideal generated by I_Y in $k[x_0, \ldots, x_N, s]$. This ideal corresponds to a variety:

$$C(Y) = \operatorname{Proj} k[x_0, \ldots, x_N, s]/I_{C(Y)} \subseteq \mathbb{P}$$

which can naturally be called the cone of Y in $\mathbb{P}$.

This relates to $\mathbf{V}(\mathscr{N}^*)$ as described in the following proposition.

Proposition 7.2. *Let Y, $\mathscr{N}$, $\mathbb{P}$, Q, and $C(Y)$ be as above. Then:*

$$C(Y) \setminus \{Q\} \simeq \mathbf{V}(\mathscr{N}^*)$$

Proof. We may cover Y by the open neighborhoods $\{U_i\}$, where

$$U_i = Y \cap \{x_i \neq 0\} \simeq \operatorname{Spec} A_i = \operatorname{Spec}((k[x_0, \ldots, x_N]/I_Y)[\overline{x_i}^{-1}])_0.$$

For $\mathscr{N} \simeq \mathscr{O}_Y(n)$, we will have $H^0(U_i, \mathscr{N}^*|_{U_i}) = A_i s_i$ with s_i corresponding to x_i^{-n} in $k[x_0, \ldots, x_N]$. Then if $\pi : \mathbf{V}(\mathscr{N}^*) \to Y$ is the bundle map, we have:

$$\begin{aligned}\pi^{-1}(U_i) &\simeq \operatorname{Spec} \bigoplus_{m=0}^{\infty} \operatorname{Sym}^m(A_i s_i) = \operatorname{Spec} A_i[s_i] \\ &= \operatorname{Spec} k[y_0, \ldots, \hat{y_i}, \ldots, y_N, s_i]/(I_Y(x_i^{-1}, s_i))_0,\end{aligned}$$

where $y_j = \frac{x_j}{x_i}$ and $\deg s_i = 0$.

Let $U_{i,j} = U_i \cap U_j$. Then:

$$\pi^{-1}(U_{i,j}) \simeq \operatorname{Spec} k[y_0, \ldots, \hat{y_i}, \ldots, y_N, y_j^{-1}, s_i]/(I_Y(x_i^{-1}, x_j^{-1}, s_i))_0,$$

given in the coordinates for $\pi^{-1}(U_i)$.

If $\{y_0, \ldots, \hat{y_i}, \ldots, y_N, s_i\}$ are the coordinates for $\pi^{-1}(U_i)$ and $\{y'_0, \ldots, \hat{y'_j}, \ldots, y'_N, s_j\}$ are the coordinate for $\pi^{-1}(U_j)$, then the transition function for gluing them is given by:

$$\begin{aligned}s_i &\leftrightarrow x_i^{-n} \mapsto y_i'^{-n} s_j \\ y_k &\leftrightarrow \frac{x_k}{x_i} \mapsto \frac{y'_k}{y'_i} \quad \text{for } k \neq i, j \\ y_j &\leftrightarrow \frac{x_j}{x_i} \mapsto y_i'^{-1}\end{aligned}$$

This completely characterizes $\mathbf{V}(\mathscr{N}^*)$. Next we will consider $C(Y) \setminus \{Q\}$.

$C(Y) \setminus \{Q\}$ is covered by the open neighborhoods $\{V_i = C(Y) \cap \{x_i \neq 0\}\}$. In fact we have:

$$\begin{aligned}V_i &\simeq \operatorname{Spec}((k[x_0, \ldots, x_N, s]/I_{C(Y)})[x_i^{-1}])_0 \\ &= \operatorname{Spec} k[y_0, \ldots, \hat{y_i}, \ldots, y_N, s_i]/(I_{C(Y)}(x_i^{-1}))_0 \\ &= \operatorname{Spec} k[y_0, \ldots, \hat{y_i}, \ldots, y_N, s_i]/(I_Y(x_i^{-1}, s_i))_0,\end{aligned}$$

where $y_j = \frac{x_j}{x_i}$ and $s_i = \frac{s}{x_i^n}$.

If $\{y_0, \ldots, \hat{y_i}, \ldots, y_N, s_i\}$ are the coordinates for V_i and $\{y'_0, \ldots, \hat{y'_j}, \ldots, y'_N, s_j\}$ are the coordinate for V_j, then the transition function on for gluing them is given by:

$$\begin{aligned}s_i &= \frac{s}{x_i^n} = y_i'^{-n} s_j \\ y_k &= \frac{x_k}{x_i} = \frac{y'_k}{y'_i} \quad \text{for } k \neq i, j \\ y_j &= \frac{x_j}{x_i} = y_i'^{-1}\end{aligned}$$

By comparing $\{V_i\}$ to $\{\pi^{-1}(U_i)\}$ along with their transition functions, it is clear that $(C(Y) \setminus \{Q\}) \simeq \mathbf{V}(\mathscr{N}^*) = \mathbf{V}(\mathscr{O}_Y(-n))$. □

We have noted that $\mathbb{P}$ is singular at Q if $n > 1$. Let $\tilde{\mathbb{P}}$ be a resolution of this singularity if $n > 1$, or the blow-up of $\mathbb{P}$ at Q if $n = 1$. Then the strict tranform of $C(Y)$ in $\tilde{\mathbb{P}}$, $\widetilde{C(Y)}$, will be a completion of $C(Y) \setminus \{Q\} \simeq \mathbf{V}(\mathscr{N}^*)$ as a $\mathbb{P}^1$-bundle over Y. According to our previous discussion, we will have $\kappa(\widetilde{C(Y)}) = -\infty$.

In the following sections we will show that for many cases when Y is a non-singular complete intersection in $\mathbb{P}^N$ of general type it is possible to deform $\widetilde{C(Y)}$ in $\tilde{\mathbb{P}}$ to a non-singular variety $\tilde{X}$, also of general type, with $\mathscr{N}_{Y|\tilde{X}} \simeq \mathscr{N} \simeq \mathscr{O}_Y(n)$.

As discussed at the end of section 6 this provides a partial answer to the second subquestion in that section. In particular, in this case the upper bound $\kappa(Y) + \mathrm{codim}_{Y|X}$ for $\kappa(X)$ can be achieved for $\mathscr{N}_{Y|X} \simeq \mathscr{O}_Y(n)$.

8. The Construction of $\tilde{X}$ when Y is a Hypersurface

Let k be an algebraically closed field of characteristic 0, and let Y be a smooth hypersurface in $\mathbb{P}^N_k$, defined by $f \in k[x_0, \dots, x_N]$, a homogeneous polynomial of degree d. For some choice of $n, m \in \mathbb{N}$ with $d > mn$, and $g \in k[x_0, \dots, x_N]$, a homogeneous polynomial of degree $d - nm$, consider the hypersurface in weighted projective space, $X \subseteq \mathbb{P}(1, \dots, 1, n) = \mathbb{P}$, defined by $f + s^m g \in k[x_0, \dots, x_N, s]$, where deg $x_i = 1$, deg $s = n$.

Y lies naturally in X as a divisor given by $s = 0$. In $\mathbb{P}(1, \dots, 1, n)$ the divisor $s = 0$ is linearly equivalent to n times the divisor $x_i = 0$ for any $0 \leq i \leq N$. Restricted to Y this is just $\mathscr{O}_Y(n)$. Thus letting $\mathscr{L}(Y)$ be the invertible sheaf on X corresponding to Y, we have:

$$\mathscr{L}(Y)|_Y = \mathscr{N}_{Y|X} = \mathscr{O}_Y(n)$$

For $n > 1$, $\mathbb{P}$ will have a singularity at $Q = (0 : \dots : 0 : 1)$. We will consider $\tilde{X}$, the strict tranform of X in $\tilde{\mathbb{P}}$, a resolution of the singularity at Q in $\mathbb{P}(1, \dots, 1, n)$ if $n > 1$ or else the blow-up of $\mathbb{P} = \mathbb{P}^{N+1}$ at Q. Since $Y \subseteq X$ is separate from $\{Q\}$, Y will still lie in $\tilde{X}$ with the same normal bundle.

To understand $\tilde{\mathbb{P}}$ we will consider $\mathbb{P}(1, \dots, 1, n)$ as a toric variety. In the following we will use the notation and results given in [1]. $\mathbb{P}(1, \dots, 1, n)$ is represented by the toric fan, Σ, in the lattice $N \simeq \mathbb{Z}^{N+1}$ generated by the one dimensional cones:

$$v_0 = \langle (1, 0, \dots, 0) \rangle, \dots, v_N = \langle (0, \dots, 0, 1) \rangle, v_{N+1} = \langle (-1, \dots, -1, -n) \rangle$$

Since $\mathbb{P}$ is toric, it is Cohen-Macaulay [1, Thm 2.1]. Since it is also regular in codimension 1, it is normal [2, II.8.22A]. This means that N-cycles on $\mathbb{P}$ with rational equivalence are equivalent to Weil divisors with linear equivalence, and since the 1-dimensional cones of Σ span $N_{\mathbb{R}}$, we have a short exact sequence:

$$0 \to M \to \bigoplus D_{v_i} \to \mathrm{Cl}\,\mathbb{P} \to 0$$

where M is the dual lattice of N and D_{v_i} represents the divisor corresponding to v_i[1, §3].

We can then compute that $D_{v_i} \sim D_{v_0}$ for $i \neq N$ and $D_{v_N} \sim nD_{v_0}$. We may let D_{v_i} corresponds to the divisor $x_i = 0$ for $0 \leq i < N$, $D_{v_{N+1}}$ correspond to $x_N = 0$, and D_{v_N} correspond to $s = 0$. As such the divisor class of X in $\mathrm{Cl}\,\mathbb{P}$ is that of dD_{v_0}.

The point $Q \in \mathbb{P}$ is the orbit corresponding to the $N+1$ dimensional cone, $\sigma = \langle v_0, \dots, v_{N-1}, v_{N+1} \rangle$ in Σ. The subfan Σ_s consisting of σ and its faces corresponds to the neighborhood, $U_s \subseteq \mathbb{P}$, defined by $s \neq 0$. If we define Σ'_s to be the same as Σ_s but on the lattice N' generated by $\{v_0, \dots, v_{N-1}, nv_N\}$, then Σ'_s will correspond to $\mathbb{A}^{N+1}$. We will have a toric morphism, $\Sigma'_s \to \Sigma_s$, corresponding to the morphism $\rho : \mathbb{A}^{N+1} \to U_s$, where ρ is the quotient map of $\mathbb{A}^{N+1}$ by the group $\mathbb{Z}/n\mathbb{Z}$ under the action: $\bar{i} \cdot (x_0, \dots, x_N) = (\zeta^i x_0, \dots, \zeta^i x_N)$, where ζ is a principal nth root of unity.

If $n > 1$, the singularity, Q, in U_s is resolved by refining the fan Σ_s with the addition of the one dimensional cone $u = \langle (0, \dots, 0, -1) \rangle$. The resolution map, $\pi : \tilde{U}_s \to U_s$, corresponds to the natural inclusion of fans. We gain an exceptional divisor, D_u. This construction corresponds in Σ'_s to blowing up $\mathbb{A}^{N+1}$ at the origin. The blow-up, $\tilde{\mathbb{A}}^{N+1}$, is also acted on by $\mathbb{Z}/n\mathbb{Z}$ in a way compatible with the action on $\mathbb{A}^{N+1}$. The resolution of the singularity and the blowing up commute with the quotient maps, so that we have a diagram:

$$\begin{array}{ccc} \tilde{\mathbb{A}}^{N+1} & \xrightarrow{\rho} & \tilde{U}_s \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{A}^{N+1} & \xrightarrow{\rho} & U_s, \end{array}$$

where the ρ are isomorphisms if $n = 1$.

We will be interested in calculating the linear equivalence class of the strict transform of X in $\tilde{\mathbb{P}}$. This result is given in the following proposition.

Proposition 8.1. *Let $X \subseteq \mathbb{P}$ be the hypersurface corresponding to $f + s^m g = 0$, where f is a homogeneous polynomial in $k[x_0, \dots, x_N]$ of degree $d > mn$ and g is one of degree $d - mn$. Let $\tilde{\mathbb{P}}$ be as described above and let $\tilde{X}$ be the divisor in $\tilde{\mathbb{P}}$ that is the strict transform of X. Then the linear equivalence class of $\tilde{X}$ can be written in terms of the torically defined divisor classes, $\{D_{v_i}\}$ and D_u, as follows:*

$$\tilde{X} \sim dD_{v_0} + mD_u$$

Proof. $\pi : \tilde{\mathbb{P}} \to \mathbb{P}$ is an isomorphism over $\mathbb{P} \setminus \{Q\}$. Thus to calculate the strict transform of X, we may work locally over U_s. For ease of notation, we will temporarily refer to $X \cap U_s$ as X and $\tilde{X} \cap \tilde{U}_s$ as $\tilde{X}$.

X is given in U_s by $f + s^m g$. This means $\rho^{-1}X$ is given by $f + g$ in $\mathbb{A}^{N+1}$, where we remember $\deg f = d$, $\deg g = d - mn$. The blow-up $\tilde{\mathbb{A}}^{N+1}$ can be seen as a subvariety of $\mathbb{A}^{N+1} \times \mathbb{P}^N$. Let the bi-coordinates of $\mathbb{A}^{N+1} \times \mathbb{P}^N$ be given in order by $x_0, \dots, x_N, y_0, \dots, y_N$. Then $\pi^*\rho^{-1}X$ is given on $\tilde{\mathbb{A}}^{N+1}$ by $f(x_0, \dots, x_N) + g(x_0, \dots, x_N)$. Since the y_i are not all zero, we can let $\lambda = \frac{x_i}{y_i}$. This gives:

$$\begin{aligned} f(x_0, \dots, x_N) + g(x_0, \dots, x_N) &= f(\lambda y_0, \dots, \lambda y_N) + g(\lambda y_0, \dots, \lambda y_N) \\ &= \lambda^d f(y_0, \dots, y_N) + \lambda^{(d-mn)} g(y_0, \dots, y_N) \\ &= \lambda^{(d-mn)}(\lambda^{mn} f(y_0, \dots, y_N) + g(y_0, \dots, y_N)) \end{aligned}$$

Thus we see the proper transform, $\pi^*\rho^{-1}X$, of $\rho^{-1}X$ breaks into components: a strict transform, $\widetilde{\rho^{-1}X}$, given by $(\lambda^{mn} f(y_0, \dots, y_N) + g(y_0, \dots, y_N))$, and $d - mn$ copies of the exceptional divisor, E, given by (λ). That is $\pi^*\rho^{-1}X = \widetilde{\rho^{-1}X} + (d - mn)E$ as divisors on $\tilde{\mathbb{A}}^{N+1}$.

Now the quotient map, $\rho : \tilde{\mathbb{A}}^{N+1} \to \tilde{U}_s$ is a degree n map of finite type, étale away from E, and taking $\widetilde{\rho^{-1}X}$ to $\tilde{X}$ with degree n and E to D_u isomorphically. Thus we have:

$$\rho_*\pi^*\rho^{-1}X = n\tilde{X} + (d - mn)D_u$$

X is not Cartier on U_s, but nX is. We have $\rho^*(nX) = n\rho^{-1}X$, so we may also write:

$$\rho_*\pi^*\rho^*(nX) = n^2\tilde{X} + n(d - mn)D_u$$

We have worked locally, but as the resolution is an isomorphism away from the exceptional locus, the above holds on $\tilde{\mathbb{P}}$ also. This makes sense because the quotient $\rho : \tilde{\mathbb{A}}^{N+1} \to \tilde{U}_s$ may be extended to the projective completion of $\tilde{\mathbb{A}}^{N+1}$ in the obvious way as $\rho : \tilde{\mathbb{P}}^{N+1} \to \tilde{\mathbb{P}}(1,\dots,1,n)$.

We will now calculate this divisor from a toric viewpoint. First observe that everything we said earlier about $\mathbb{P}$ also holds for $\tilde{\mathbb{P}}$, so that we have a similar short exact sequence useful for calculating divisor classes. We have $nX \sim ndD_{v_0}$. ndD_{v_0} is given on the cone $\langle v_0,\dots,v_{N-1},v_{N+1}\rangle$ by a linear operator, χ, on N with $\chi(v_0) = nd$ and $\chi(v_i) = 0$ for $i \neq 0$. Thus χ is given in the dual lattice, M, by the vector $(nd, 0,\dots,0,-d)$. Then $\chi(u) = d$ shows that $\pi^*(ndD_{v_0}) \sim ndD_{v_0} + dD_u$.

Since the quotient map $\rho : \tilde{\mathbb{P}}^{N+1} \to \tilde{\mathbb{P}}$ is of finite type and degree n, we have:

$$\rho_*\rho^*\pi^*(nX) \sim \rho_*\rho^*\pi^*(ndD_{v_0}) \sim n^2dD_{v_0} + ndD_u$$

Observing that $\pi \circ \rho = \rho \circ \pi$, we may combine this with our previous calculation to conclude:

$$n^2\tilde{X} + n(d - mn)D_u \sim n^2dD_{v_0} + ndD_u$$

$$\implies \tilde{X} \sim dD_{v_0} + mD_u$$

□

9. The Construction of $\tilde{X}$ when Y is a Complete Intersection

We now want to consider the more general case when Y is a complete intersection in $\mathbb{P}^N$. Let $Y \subseteq \mathbb{P}^N$ be a non-singular projective variety of dimension $N - r$ given as the intersection of the hypersurfaces $f_t = 0,\ 1 \leq t \leq r$, where $f_t \in k[x_0,\dots,x_N]$ with $\deg f_t = d_t > 1$.

We will construct a variety of dimension $N - r + 1$ containing Y. In particular, let X_t be the hypersurface in $\mathbb{P}(1,\dots,1,n)$ given by $f_t + s^{m_t}g_t = 0$, where either $1 \leq m_t < \frac{d_t}{n}$ and $g_t \in k[x_0,\dots,x_N]$ is homogeneous of degree $d_t - m_tn$, or $m_t = 0$ and $g_t = 0$. Let $\tilde{X}_t \subseteq \tilde{\mathbb{P}}$ be the strict transform of X_t. From our work in the last section we know that as a divisor:

$$\tilde{X}_t \sim d_tD_{v_0} + m_tD_u$$

Let $X = \bigcap X_t$. Then $Y \simeq X \cap \{s = 0\}$. This means Y lies in X with $\mathscr{N}_{Y|X} \simeq \mathscr{O}_Y(n)$. Let $\tilde{X} \subseteq \tilde{\mathbb{P}}$ be the strict tranform of X. Then $\tilde{X} = \bigcap \tilde{X}_t$ and Y will also lie in $\tilde{X}$ with $\mathscr{O}_Y(n)$ as a normal bundle.

If $\tilde{X}$ is non-singular we will have:

$$K_{\tilde{X}} \sim (K_{\tilde{\mathbb{P}}} + \sum \tilde{X}_t)|_{\tilde{X}}$$

We know from the theory of toric varieties that:

$$K_{\tilde{\mathbb{P}}} \sim -\sum D_{v_i} - D_u \sim (-1 - n - N)D_{v_0} - 2D_u$$

since $D_{v_N} \sim nD_{v_0} + D_u$.

Combining this with the previous results shows:

Proposition 9.1. *If $\tilde{X}$ is non-singular, then $K_{\tilde{X}} \sim ((d - 1 - n - N)D_{v_0} + (m - 2)D_u)|_{\tilde{X}}$, where $d = \sum d_t$ and $m = \sum m_t$.*

10. When is $\tilde{X}$ Non-singular?

Next we will want to show that for almost all $\{g_t\}$, $\tilde{X}$ is non-singular. We will use the following proposition.

Proposition 10.1. *Let $f \in k[x_0, \dots, x_N]$ be a homogeneous polynomial of degree $d > n$, let $1 \leq m < \frac{d}{n}$, and and let $U = \mathbb{A}^{N+1} \setminus \{f = 0\}$. Then the linear system, $\mathfrak{d}$, defined by $\{af + g\} \subseteq H^0(U, \mathscr{O}_U)$ for any $a \in k$, $g \in k[x_0 \dots, x_N]$ with g homogeneous of degree $d - mn$ determines a morphism, $U \xrightarrow{\rho} U' \xhookrightarrow{i} \mathbb{P}^L$, where ρ is the quotient by $\mathbb{Z}/mn\mathbb{Z}$, i is an embedding, and $L = \binom{N+d-mn}{N}$.*

Proof. It is clear that $\mathfrak{d}$ is a linear system on U with the correct dimension. Thus we need to show that $\mathfrak{d}$ has no base points on U, does not distinguish between points in the same fiber of ρ, and separates points and tangents on U'.

Since f is non-zero for every point in U, it is clear that $\mathfrak{d}$ has no base points. Let $P, Q \in U$ with $\rho(P) = \rho(Q)$. This means $Q = \zeta P$ for $\zeta \in k$, $\zeta^{mn} = 1$. Then, if $(af + g)(P) = 0$, we have:

$$(af + g)(Q) = af(\zeta P) + g(\zeta P) = \zeta^d af(P) + \zeta^{d-mn} g(P) = \zeta^d (af(P) + g(P)) = 0$$

Thus $U \xrightarrow{\mathfrak{d}} \mathbb{P}^L$ factors through U'.

Now let $P, Q \in U$ with $\rho(P) \neq \rho(Q)$ If Q is not a multiple of P, that is, not on the same line through the origin, then we can choose g so that $g(Q) = 0$, but $g(P) \neq 0$. Let $a = -\frac{g(P)}{f(P)}$, then $af + g \in \mathfrak{d}$ is zero at P, but non-zero at Q. If $Q = \alpha P$ with $\alpha^{mn} \neq 1$, choose g so that $g(Q) = \alpha^{d-mn} g(P) \neq 0$, and choose $a = -\frac{g(P)}{f(P)}$ so that $(af + g)(P) = 0$, but:

$$(af + g)(Q) = af(\alpha P) + g(\alpha P) = \alpha^d af(P) + \alpha^{d-mn} g(P)$$
$$= (\alpha^d - \alpha^{d-mn}) af(P) + \alpha^{d-mn}(af(P) + g(P)) = \alpha^{d-mn}(\alpha^{mn} - 1) af(P) \neq 0$$

This shows that $\mathfrak{d}$ separates the points of U'.

Now we will show that $\mathfrak{d}$ separates tangents on U. Let $P = (x_0, \dots, x_N) \in U$ and $\mathbf{v} = \langle v_0, \dots, v_N \rangle \in \mathscr{T}_{U,P}$ be a non-zero element of the tangent space of U at P. We want to choose $af + g$ so that $(af + g)(P) = 0$ and $\mathbf{v}$ is not in the tangent space of $af + g = 0$, that is $\nabla(af + g)(P) \cdot \mathbf{v} \neq 0$.

If $\mathbf{v} \nparallel \langle x_0, \dots, x_N \rangle$, then let $a = 0$. We can choose g so that $g(P) = 0$ with $\nabla g(P)$ arbitrary as long as $\nabla g(P) \cdot \langle x_0, \dots, x_N \rangle = 0$. In particular we can choose g with $g(P) = 0$ and $\nabla g(P) \cdot \mathbf{v} \neq 0$.

Now suppose $\mathbf{v} \parallel \langle x_0, \dots, x_N \rangle$. Choose g so that $g(P) \neq 0$. By changing coordinates if necessary, we may assume $P = (1, 0, \dots, 0)$ and $\mathbf{v} = \langle \alpha, 0, \dots, 0 \rangle$. Then if:

$$f = a_0 x_0^d + \text{ terms of lower degree in } x_0,$$

we have $f(P) = a_0$, and

$$\nabla f(P) \cdot \mathbf{v} = \alpha \frac{\partial f}{\partial x_0}(P) = \alpha d a_0$$

In particular, $\nabla f(P) \cdot \mathbf{v} = \alpha d(f(P))$. Similarly $\nabla g(P) \cdot \mathbf{v} = \alpha(d - mn) g(P)$. Thus if we again let $a = -\frac{g(P)}{f(P)}$, we have $(af + g)(P) = 0$, and:

$$\nabla(af + g)(P) \cdot \mathbf{v} = a \nabla f(P) \cdot \mathbf{v} + \nabla g(P) \cdot \mathbf{v}$$
$$= a\alpha d(f(P)) + \alpha(d - mn) g(P) = -\alpha mn g(P) \neq 0$$

This means $\mathfrak{d}$ separates tangents of U. Since locally U' looks like U, $\mathfrak{d}$ will also separate tangents on U', which proves the proposition. □

Now we will present the main result of this section, which shows that $\tilde{X}$ can be constructed to be non-singular, so that we may apply proposition 9.1 in order to determine its Kodaira dimension.

Proposition 10.2. *Let Y be a non-singular complete intersection in $\mathbb{P}^N$ of type $(d_1, \ldots, d_r)$. For some appropriate choice of $(m_1, \ldots, m_r)$, let $\tilde{X}$ be defined as above. Then except for a codimension 1 set in the space of possible g_t, $\tilde{X}$ will be non-singular.*

Proof. We will proceed by induction on r. If $r = 0$, we take $Y = \mathbb{P}^N$ and $\tilde{X} = \tilde{\mathbb{P}}$, which is non-singular. If $m_t = 0$ for all t, then we have required $g_t = 0$ for all t. In this case X is defined in $\mathbb{P}(1, \ldots, 1, n)$ by $f_1 = \ldots = f_r = 0$. That is, X is the cone over Y in $\mathbb{P}$, and $\tilde{X}$ is its strict transform in $\tilde{\mathbb{P}}$. Then $\tilde{X}$ will be non-singular since Y is.

Thus we assume not all m_i are zero. Rearrange the indices if necessary so that $m_r \neq 0$. We will let

$$Z = \{f_1 + s^{m_1} g_1 = \ldots = f_{r-1} + s^{m_{r-1}} g_{r-1} = 0\} \subseteq \mathbb{P},$$

and let $\tilde{Z}$ be its strict transform in $\tilde{P}$. Then by the inductive hypothesis $\tilde{Z}$ is non-singular for almost all g_t, and $X = Z \cap \{f_r + s^{m_r} g_r = 0\}$.

We will show in the following four lemmas that $\tilde{X}$ is non-singular in four regions:

$$\tilde{X} \cap \{s = 0\}, \quad \tilde{X} \cap D_u, \quad \tilde{X} \cap (\tilde{U_s} \setminus (D_u \cup \bigcup_{m_t \neq 0} \{f_t = 0\})),$$

$$\text{and} \quad \tilde{X} \cap (\tilde{U_s} \setminus D_u) \cap \bigcup_{m_t \neq 0} \{f_t = 0\}$$

By the localness of non-singularity this will demonstrate that $\tilde{X}$ is non-singular. □

Lemma 10.3. *$\tilde{X}$ is non-singular at all closed points in $\tilde{X} \cap \{s = 0\}$.*

Proof. $\tilde{X} \cap \{s = 0\} \subseteq \mathbb{P}$ can be covered by the $N + 1$ neighborhoods of the form $U_i = \{x_i \neq 0\}$ with coordinates $\{y_j = \frac{x_j}{x_i}, s' = \frac{s}{x_i^n}\}$ for $j \neq i$. Let $f + s^m g = 0$ on U_i correspond to $f'(y_0, \ldots, \hat{y}_i, \ldots, y_N) + s'^m g'(y_0, \ldots, \hat{y}_i, \ldots, y_N) = 0$. Then X will be non-singular for $s = 0$ if the cotangent vectors, $\{\nabla(f_1' + s'^{m_1} g_1'), \ldots, \nabla(f_r' + s'^{m_r} g_r')\}$ are linearly independents for all points in $X \cap U_i \cap \{s = 0\}$ for all i, where ∇ is the gradient operator:

$$\nabla = \left\langle \frac{\partial}{\partial y_0}, \ldots, \widehat{\frac{\partial}{\partial y_i}}, \ldots, \frac{\partial}{\partial y_N}, \frac{\partial}{\partial s'} \right\rangle$$

For $s = 0 \implies s' = 0$, we have:

$$\frac{\partial}{\partial y_j}(f' + s'^m g') = \frac{\partial f'}{\partial y_j} + s'^m \frac{\partial g'}{\partial y_j} = \frac{\partial f'}{\partial y_j},$$

since either $m > 0$ or $g = 0$. Now:

$$\nabla(f' + s'^m g') = \left\langle \frac{\partial f'}{\partial y_0}, \ldots, \widehat{\frac{\partial f'}{\partial y_i}}, \ldots, \frac{\partial f'}{\partial y_N}, \frac{\partial}{\partial s'}(f' + s'^m g') \right\rangle$$

We know that Y is non-singular. This means the vectors $\{\langle \frac{\partial f'_t}{\partial y_0}, \ldots, \widehat{\frac{\partial f'_t}{\partial y_i}}, \ldots, \frac{\partial f'_t}{\partial y_N} \rangle\}$ are linearly independent for $1 \le t \le r$. But this means the vectors $\{\nabla(f'_t + s'^{m_t} g_t)\}$ are linearly independent. So X is non-singular on $X \cap \{s = 0\}$ for any choice of g_t. The same holds for $\tilde{X}$ since the resolution is an isomorphism in a neighborhood of $\{s = 0\}$. □

Lemma 10.4. *$\tilde{X}$ is non-singular at all closed points in $\tilde{X} \cap D_u$ for all $\{g_t \in k[x_0, \ldots, x_N]_{(d_t - m_t n)}\}$ outside a set of codimension 1.*

Proof. By considering the toric description of $\tilde{\mathbb{P}}$, we see that the exceptional set of the resolution, $D_u \subseteq \tilde{\mathbb{P}}$, is covered by $N+1$ affine neighborhoods:

$$\{U_i \simeq \operatorname{Spec} k\left[\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_N}{x_i}, \frac{x_i^n}{s}\right]\}$$

Let us call the local coordinates of U_i, $(y_0, \ldots, \hat{y}_i, \ldots, y_N, v)$. Then the strict transform of $\{f + s^m g = 0\}$ is given in U_i by $v^m f' + g' = 0$. $\tilde{X}$ will be non-singular at the points of $\tilde{X} \cap D_u$ if the vectors $\{\nabla(v^{m_t} f'_t + g'_t)\}$ are linearly independent at all points of $\tilde{X}$ where $v = 0$ for all i. By the analysis in the proof of the last lemma, we see this is true if the vectors $\{\nabla g'_t$ (or $\nabla f'_t$ if $m_t = 0)\}$ are linearly independent for all points of $\tilde{X} \cap D_u \subseteq D_u \simeq \mathbb{P}^N$.

We assume by induction that $\tilde{Z} \cap D_u \subseteq D_u$ is non-singular for almost all $\{g_1, \ldots, g_{r-1}\}$. Then we want $(\tilde{Z} \cap D_u) \cap \{g_t = 0\}$ to be non-singular. This will be true if for every point $P \in \tilde{Z} \cap D_u$ we do not have $g_r(P) = 0$ and $\nabla g_r \in \mathscr{N}_{\tilde{Z} \cap D_u, P}$, that is ∇g_r is not in the normal space of $\tilde{Z} \cap D_u$. This space has dimension $r - 1$, so the condition we wish to avoid is a condition of codimension $(N - (r-1)) + 1 = N - r + 2$ for $g_r \in k[x_0, \ldots, x_N]_{(d_r - m_r n)}$. Since $\tilde{Z} \cap D_u$ has dimension $N - r + 1$, we see that for g_r chosen outside a set of codimension 1 in $k[x_0, \ldots, x_N]_{(d_r - m_r n)}$, we avoid the given condition for all P.

Thus for almost all $\{g_t\}$, $\tilde{X}$ will be non-singular for the points of $\tilde{X} \cap D_u$. □

Lemma 10.5. *$\tilde{X}$ is non-singular at all closed points in*

$$\tilde{X} \cap (\tilde{U}_s \setminus (D_u \cup \bigcup_{m_t \neq 0} \{f_t = 0\}))$$

for all $\{g_t \in k[x_0, \ldots, x_N]_{(d_t - m_t n)}\}$ outside a set of codimension 1.

Proof. By induction we know $\tilde{Z}$ is non-singular in $\tilde{Z} \cap \tilde{U}_s$ for almost all choices of $\{g_t\}$. Let $W = \rho^{-1}(Z \cap U_s)$ be the inverse image of $Z \cap U_s$ under the quotient map $\rho : \mathbb{A}^{N+1} \to U_s$. Then W will be non-singular away from the origin, and:

$$W = \bigcap_{t=1}^{r-1} \{f_t + g_t = 0\}$$

Now if $(f_t + g_t)(P) = 0$ and $f_t(P) \neq 0$ for $P \in \mathbb{A}^{N+1}$, then it is easy to calculate that $(f_t + g_t)(\zeta P) = 0$ if and only if $\zeta^{m_t n} = 1$. On the otherhand if $m_t = 0$, then $f_t(P) = 0$ implies $f_t(\alpha P) = 0$ for all $\alpha \in k$.

From this we conclude that the quotient of $\mathbb{A}^{N+1}$ by the action of $\mathbb{Z}/m_r n\mathbb{Z}$ restricted to $(W \setminus \bigcup_{m_t \neq 0} \{f_t = 0\})$ is an étale covering of degree $\gcd(\{m_t\})n$.

If $\mathfrak{d}$ is the linear system described in proposition 10.1 defined by $\{af_r + g\}$ with $\deg g = d_r - m_r n$, then the map defined by $\mathfrak{d}$ factors as $\mathbb{A}^{N+1} \setminus \{f_r = 0\} \to U' \longrightarrow$

$\mathbb{P}^L$. When restricted to $(W \setminus \bigcup_{m_t \neq 0}\{f_t = 0\})$ the first map is the étale covering discussed. Then we have:

$$(W \setminus \bigcup_{m_t \neq 0} \{f_t = 0\}) \xrightarrow{\rho} W' \xrightarrow{i} \mathbb{P}^L$$

The last map is an embedding according to proposition 10.1. Now Bertini's theorem tells us that for almost any section of $\mathfrak{d}$, $af_r + g$ (we may take $a \neq 0$), the corresponding hyperplane section $i(W') \cap H$ is non-singular. This means $i^{-1}(H)$ is non-singular, and since ρ is étale, $\rho^{-1}i^{-1}(H)$ is non-singular. However,

$$\rho^{-1}i^{-1}(H) = (W \setminus \bigcup_{m_t \neq 0} \{f_t = 0\}) \cap \{af_r + g = 0\}$$
$$= (W \setminus \bigcup_{m_t \neq 0} \{f_t = 0\}) \cap \{f_r + \frac{1}{a}g = 0\}$$

This variety maps according to the quotient by $\mathbb{Z}/n\mathbb{Z}$ in an étale fashion to U_s to give:

$$Z \cap (U_s \setminus \bigcup_{m_t \neq 0} \{f_t = 0\}) \cap \{f_r + s^{m_r}\frac{1}{a}g = 0\}$$

If we take $g_r = \frac{1}{a}g$, this is just equal to:

$$X \cap (U_s \setminus \bigcup_{m_t \neq 0} \{f_t = 0\})$$

Since the resolution $\tilde{X} \to X$ is an isomorphism away from $Q \in U_s$, the lemma follows. $\square$

Lemma 10.6. *$\tilde{X}$ is non-singular at all closed points in*

$$\tilde{X} \cap (\tilde{U}_s \setminus D_u) \cap \bigcup_{m_t \neq 0} \{f_t = 0\}$$

for all $\{g_t \in k[x_0, \ldots, x_N]_{(d_t - m_t n)}\}$ outside a set of codimension 1.

Proof. The argument here is similar to the one in lemma 10.4. By induction we know that Z is non-singular on $U_s \setminus \{Q\}$ for almost all $\{g_t\}$. Let $W = \rho^{-1}(Z \cap U_s)$ as in the last lemma. Then for $\rho^{-1}(X \cap U_s)$ to be non-singular on $(\rho^{-1}(X \cap U_s) \setminus \{(0, \ldots, 0)\}) \cap \bigcup_{m_t \neq 0}\{f_t = 0\}$, we need to avoid for all points $P \in (W \setminus \{(0, \ldots, 0)\}) \cap \bigcup_{m_t \neq 0}\{f_t = 0\}$ the condition:

$$g_r(P) = -f_r(P) \quad \text{and} \quad \nabla g_r + \nabla f_r \in \mathscr{N}_{W,P}$$

Fixing $g_r(P)$ restricts ∇g_r to a N-dimensional space. Since codim $W = r - 1$, dim $\mathscr{N}_{W,P} = r - 1$. This mean the above condition has codimension $(N - (r-1)) + 1 = N - r + 2$ in the space, $k[x_0, \ldots, x_N]_{(d_r - m_r n)}$, of possible g_r. Now:

$$\dim\left((W \setminus \{(0, \ldots, 0)\}) \cap \bigcup_{m_t \neq 0} \{f_t = 0\}\right) = (N+1) - r.$$

Thus taking the union of our sets to be avoided over all points of interest gives a set of codimension 1 in the space of g_r outside of which $\rho^{-1}(X \cap U_s)$ is non-singular on $(\rho^{-1}(X \cap U_s) \setminus \{(0, \ldots, 0)\}) \cap \bigcup_{m_t \neq 0}\{f_t = 0\}$. Since ρ is étale and π is an isomorphism over $U_s \setminus \{Q\}$, this means that $\tilde{X}$ is non-singular in $\tilde{X} \cap (\tilde{U}_s \setminus D_u) \cap \bigcup_{m_t \neq 0}\{f_t = 0\}$. $\square$

Now looking at the results of the last four lemmas and considering that non-singularity is a local condition shows that outside a codimension 1 set in the space of $\{g_t\}$ we have that $\tilde{X}$ is non-singular as stated in the proposition. □

11. The Kodaira Dimension of $\tilde{X}$

Having established that for almost all choices of $\{g_t\}$ $\tilde{X}$ is non-singular, we now want to say something about $\kappa(\tilde{X})$ in order to address the first subquestion in section 6 that we were interested in. We start with the following proposition.

Proposition 11.1. *Let $d = \sum d_t$ and $m = \sum m_t$. If $\tilde{X}$ is non-singular and $m = 2$, then $\tilde{X}$ is of general type if and only if $d > N + n + 1$.*

Proof. From the formula for $K_{\tilde{X}}$ in proposition 9.1 we see that if $d < N + n + 1$ then $K_{\tilde{X}}$ is anti-effective, so $\kappa(\tilde{X}) = -\infty$. If $d = N + n + 1$ then $K_{\tilde{X}} \sim 0$, so $\kappa(\tilde{X}) = 0$.

Now suppose $d > N + n + 1$. Then:

$$(n+1)K_{\tilde{X}} \sim aD_{v_0}|_{\tilde{X}} \text{ for } a = (n+1)(d-1-n-N) \geq n+1.$$

Let D be a divisor equivalent to aD_{v_0} in $\mathbb{P}$ passing through Q. Then by the same technique used to determine the equivalence class of $\tilde{X}$, we find that $\tilde{D} \sim aD_{v_0}$, as divisors in $\tilde{\mathbb{P}}$, where $\tilde{D}$ is the strict transform of D.

Thus there is an injective map, $\mathfrak{d} \hookrightarrow |(n+1)K_{\tilde{X}}|$, where $\mathfrak{d}$ is the linear subsystem of $|aD_{v_0}|$ of divisors passing through Q in $\mathbb{P}$. As a subsystem of $\mathfrak{d}$ we have the linear system $\mathfrak{d}_f$ defined by $\{af+sg\}$ for a fixed non-zero $f \in k[x_0, \ldots, x_N]_a$ and arbitrary $a \in k$ and $g \in k[x_0, \ldots, x_N]_{(a-n)}$. According to proposition 10.1 this induces a map on $U_s \setminus \{Q\}$ whose image has full dimension. Thus we have:

$$\kappa(\tilde{X}) \geq \dim\, \varphi_{(n+1)K_{\tilde{X}}}(\tilde{X}) \geq \dim\, \varphi_{\mathfrak{d}_f}(X) = \dim\, X$$

We must have $\kappa(\tilde{X}) = \dim\, \tilde{X}$. Thus $\tilde{X}$ is of general type. □

Finally we may write down a proposition giving sufficient conditions for a complete intersection projective variety to lie as a hypersurface in a variety of general type or of Kodaira dimension 0 with normal bundle $\mathscr{O}(n)$ for some n. This is a slight expansion of the previously stated proposition 6.2.

Proposition 11.2. *Let $Y \subseteq \mathbb{P}^N$ be a non-singular complete intersection variety of multi-degree $(d_1, \ldots, d_r)$ with $d_t > 1$ for all t, and $n \in \mathbb{N}$. If $d_i > 2n$ for some i, or $d_i, d_j > n$ for some $i \neq j$, then Y lies in a non-singular projective variety $\tilde{X}$ with $\mathscr{N}_{Y|\tilde{X}} \simeq \mathscr{O}_Y(n)$, where $\tilde{X}$ is of general type if $\sum d_t > N + n + 1$ and $\kappa(\tilde{X}) = 0$ if $\sum d_t = N + n + 1$.*

Proof. If $d_i > 2n$, let $m_i = 2$ and $m_t = 0$ for all $t \neq i$. If $d_i, d_j > n$, let $m_i = m_j = 1$ and $m_t = 0$ for all $t \neq i, j$. Now construct $\tilde{X}$ as in section 9 for the given $\{m_t\}$. We have $\mathscr{N}_{Y|\tilde{X}} \simeq \mathscr{O}_Y(n)$. By proposition 10.2 we may choose $\{g_t\}$ so that $\tilde{X}$ is non-singular. $\tilde{X}$ is the resolution or the blow-up of a complete intersection in $\mathbb{P}$; therefore $\tilde{X}$ is projective.

Since $\sum m_t = 2$, we may apply proposition 11.1 and its proof to conclude that $\tilde{X}$ is of general type if $\sum d_t > N + n + 1$ and $\kappa(\tilde{X}) = 0$ if $\sum d_t = N + n + 1$. □

References

[1] David A. Cox. Toric varieties and toric resolutions. *Progress in Mathematics*, 181:259–284, 2000.

[2] Robin Hartshorne. *Algebraic Geometry*. Springer-Verlag, 1977.

[3] Shigeru Iitaka. *Algebraic Geometry: An Introduction to Birational Geometry of Algebraic Varieties*. Springer-Verlag, 1981.

[4] Shigefumi Mori. Classification of higher-dimensional varieties. *Proceedings of Symposia in Pure Mathematics*, 46, 1987.

[5] Thomas Peternell, Michael Schneider, and Andrew J. Sommese. Kodaira dimension of subvarieties. *International Journal of Mathematics*, 10(8):1065–1079, 1999.