# TORELLI THEOREM FOR ALGEBRAIC CURVES

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## 1. Introduction

The Torelli theorem for algebraic curves is a result showing that an algebraic curve or equivalently a Riemannian surface can be recovered up to isomorphism from its polarized Jacobian. We will thus spend some time discussing the construction of this Jacobian. In particular we will show that the Jacobian of a curve inherits a polarized Hodge structure from the curve, which has an essentially trivial one. We will start by defining polarized Hodge structure.

## 2. Polarization

Given a Kahler manifold $X$, $H = H^n(X, \mathbb{C})$ has a Hodge decomposition as

$$H = \bigoplus_{p+q=n} H^{p,q}, \qquad H^{p,q} = \overline{H^{q,p}}$$

This is a Hodge structure of weight $n$ and is said to be polarized if, in addition, there is a bilinear form $Q : H \times H \to \mathbb{C}$ satisfying the following:

$$Q(\phi, \psi) = (-1)^n Q(\psi, \phi)$$

$$Q(\phi, \psi) = 0, \quad for\ \phi \in H^{p,q},\ \psi \in H^{p',q'},\ p \neq q'$$

$$i^{p-q} Q(\phi, \overline{\phi}) > 0, \quad for\ \phi \in H^{p,q},\ \phi \neq 0$$

One natural way to obtain a polarized Hodge structure is to be given a complex manifold, $X$, with a closed form, $\omega$, of type (1,1) whose cohomology class is the Chern class of a positive line bundle, $L$, on $X$. That is:

$$[\omega] = c_1(L)$$

The pair $(X, \omega)$ is called a polarized algebraic variety. And in this case we can define a bilinear form, $Q$, by:

$$Q(\phi, \psi) = (-1)^{n+1} \int_X \phi \wedge \psi \wedge \omega^k$$

$$\phi,\ \psi \in H^n(X, \mathbb{C}) \quad k = dim_{\mathbb{C}} X - n$$

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This bilinear form on $H^n(X,\mathbb{C})$ defines a polarized Hodge structure.

## 3. Algebraic Curves

Consider a Riemannian surface, $X$, of genus $g$. This means $dimH^1(X) = 2g$. We may choose a basis, $\{\eta_1,\dots,\eta_g,\mu_1,\dots,\mu_g\}$ for $H^1(X,\mathbb{Z})$ so that:

$$\int_X \eta_i \wedge \eta_j = \int_X \mu_i \wedge \mu_j = 0$$

$$\int_X \eta_i \wedge \mu_j = \delta_{ij}$$

where $\delta_{ij}$ is the kronecker delta.

This corresponds to a dual basis, $\{\eta_1^*,\dots,\eta_g^*,\mu_1^*,\dots,\mu_g^*\}$ for $H_1(X,\mathbb{Z})$ with intersection matrix, $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$

Let $\{\omega_1,\dots,\omega_g\}$ be a basis for $H^{1,0} \subseteq H^1(X,\mathbb{C})$. We may normalize this basis by requiring $\omega_i(\eta_j^*) = \delta_{ij}$. Then, written in the basis for $H^1(X)$ given above, we have:

$$\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} = \begin{pmatrix} I & Z \end{pmatrix}$$

$$\text{or } \omega_i = \eta_i + \sum z_{ij}\mu_j$$

for some $g \times g$ matrix, $Z = (z_{ij})$. This is called the period matrix for $X$ with this choice of bases for $H^1(X,\mathbb{Z})$ and $H^{1,0}(X,\mathbb{C})$.

Since $dim_{\mathbb{C}}X = 1$ we see from the last section that there is a natural bilinear form, $Q : H^1 \times H^1 \to \mathbb{C}$, independent of any polarizing form. This gives a polarized Hodge structures. By the axioms of polarized Hodge structures we have:

$$Q(\omega_i,\omega_j) = \int_X \omega_i \wedge \omega_j = z_{ji} - z_{ij} = 0$$

$$iQ(\omega_i,\overline{\omega_i}) = i\int_X \omega_i \wedge \overline{\omega_i} = i(\overline{z_{ii}} - z_{ii}) = Im\ z_{ii} > 0$$

This means $Z$ is symmetric ($Z^t = Z$) and $Im\ Z$ is positive definite. These properties hold for all period matrices. This set of matrices, $Z \in M(g,\mathbb{C})$, with the above properties is called the Siegel half plane $H_g$.

## 4. The Jacobian

Given an algebraic curve, $X$, and a basis, $\{\omega_1, \dots, \omega_g\}$, for $H^{1,0}(X, \mathbb{C})$, choose a basis, $\{\eta_1^*, \dots, \eta_g^*, \mu_1^*, \dots, \mu_g^*\}$, for $H_1(X, \mathbb{Z})$ as above. Then we define the Jacobian of the curve to be:

$$\mathcal{J}(X) = \mathbb{C}^g / \Lambda$$

where $\Lambda$ is a $\mathbb{Z}$-lattice generated by $\{(\int_{\eta_i^*} \omega_1, \dots, \int_{\eta_i^*} \omega_g)\}, \{(\int_{\mu_i^*} \omega_1, \dots, \int_{\mu_i^*} \omega_g)\}$.

Notice that $H_1(\mathcal{J}(X), \mathbb{Z}) \simeq H_1(X, \mathbb{Z})$, and that $\mathcal{J}(X)$ is independent of the choice of basis for $H_1(X, \mathbb{Z})$. Also $\mathcal{J}(X)$ has the real topology of a torus. This means $H^2(\mathcal{J}(X), \mathbb{Z}) \simeq Hom(\bigwedge^2 H_1(\mathcal{J}(X), \mathbb{Z}), \mathbb{Z})$. So the intersection form for $H_1(X, \mathbb{Z}) \simeq H_1(\mathcal{J}(X), \mathbb{Z})$ gives a differential form, $[\omega_s] \in H^2(\mathcal{J}(X), \mathbb{Z})$.

We will show that this form is a polarizing form by showing that the corresponding bilinear form, $Q : H^1(\mathcal{J}(X), \mathbb{C}) \times H^1(\mathcal{J}(X), \mathbb{C}) \to \mathbb{C}$, meets the axioms for a polarized Hodge structure.

First we will calculate what $\Lambda$ is in the $\mathbb{C}^g$ coordinates. We have from above that $\omega_i = \eta_i + \sum z_{ij}\mu_j$. Thus:

$$\int_{\eta_i^*} \omega_j = \delta_{ij} \qquad \int_{\mu_i^*} \omega_j = z_{ji}$$

So the matrix form of $\Lambda$ is $\begin{pmatrix} I & Z \end{pmatrix}$, where the columns represent elements of $\mathbb{C}^g$.

If we let $\{dz_i\}$ be the natural basis for $H^{1,0}(\mathcal{J}(X), \mathbb{C})$ and $\{dx_i\}$, $\{dy_j\}$ be the differentials dual to $\{\eta_i^*\}$, $\{\mu_j^*\}$ respectively, then we have:

$$dz_i = dx_i + \sum z_{ij} dy_j$$

Also we may choose as a representative of the intersection form:

$$\omega_s = \sum \delta_{ij} dx_i \wedge dy_j$$

Our bilinear form is given by:

$$Q(\phi, \psi) = \int_X \phi \wedge \psi \wedge \omega_s^{g-1}$$

Normalizing, we may assume:

$$\int_X dx_1 \wedge dy_1 \wedge \ldots \wedge dx_g \wedge dy_g = 1$$

This means:

$$Q(dx_i, dx_j) = Q(dy_i, dy_j) = 0$$
$$Q(dx_i, dy_j) = \delta_{ij}(g-1)!$$

Now we wish to calculate $Q$ for the Hodge bases for $H^{1,0}(\mathcal{J}(X), \mathbb{C})$ and $H^{0,1}(\mathcal{J}(X), \mathbb{C})$. If we remember $Z$ is in the Siegel half plane, that is $Z^t = Z$ and $ImZ$ is positive definite, then we find:

$$Q(dz_i, dz_j) = (g-1)!(z_{ji} - z_{ij}) = 0$$
$$Q(dz_i, \overline{dz_j}) = (g-1)!(\overline{z_{ji}} - z_{ij})$$
$$\implies iQ(dz_i, \overline{dz_i}) = i(g-1)!(\overline{z_{ii}} - z_{ii}) = (g-1)! Imz_{ii} > 0$$

Thus the Hodge decomposition of $H^1(\mathcal{J}(X), \mathbb{C})$ along with $Q$ gives a polarized Hodge structure. In fact it is the same structure that we found on $H^1(X, \mathbb{C})$. So we may say that $\mathcal{J}(X)$ inherits a polarized Hodge structure from $X$, which is a result of $Z$ being in the Siegel half plane.

The polarized Hodge structure on $\mathcal{J}(X)$ corresponds to the form $[\omega_s] \in H^2(\mathcal{J}(X), \mathbb{Z})$, which is thus a polarizing form. We say $(\mathcal{J}(X), \omega_s)$ is the polarized Jacobian of $X$.

## 5. Automorphisms of the Jacobian

An isomorphism between two polarized Kahler manifolds, $(X, \omega)$ $(X', \omega')$ is given by an isomorphism of complex manifolds, $\varphi : X \to X'$ such that $\varphi^* \omega' = \omega$.

In particular given an algebraic curve, $X$, with polarized Jacobian, $(\mathcal{J}(X), \omega_s)$, $\mathcal{J}(X) = \mathbb{C}^g / \Lambda$, an automorphism of $\mathcal{J}(X)$ is given by a $\mathbb{C}$-linear map, $\varphi : \mathbb{C}^g \to \mathbb{C}^g$ such that $\varphi(\Lambda) = \Lambda$ and $\varphi^* \omega_s = \omega_s$.

In order to write this out explicitly we will consider a coordinate basis for $\mathbb{C}^g$ coming from $H_1(\mathcal{J}(X), \mathbb{Z})$. Remember that above we chose $\{dx_1, \ldots, dx_g, dy_1, \ldots, dy_g\}$ to be a dual basis to $\{\eta_1^*, \ldots, \eta_g^*, \mu_1^*, \ldots, \mu_g^*\}$. Let the corresponding real coordinates be $\{x_1, \ldots, x_g, y_1, \ldots, y_g\}$. Then we may represent points and vectors in $\mathbb{C}^g$ as $2g$ row vectors, and represent the linear map, $\varphi$, by a $2g \times 2g$ matrix, $\gamma$, which acts on such row vectors from the right. Thus for $\alpha \in \mathbb{C}^g$, we have $\alpha\gamma = \varphi(\alpha)$.

In this system $\Lambda = I$, and the requirement that $\varphi(\Lambda) = \Lambda$ is equivalent to $\gamma \in SL(2g, \mathbb{Z})$.

$\omega_s$ acts by the intersection matrix. With these coordinates we have:

$$\omega_s(\alpha, \beta) = \alpha \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \beta^t$$

for $\alpha, \beta \in T(\mathcal{J}(X))$.

Then the requirement $\varphi^* \omega_s = \omega_s$ becomes:

$$\omega_s(\alpha, \beta) = \omega_s(\alpha\gamma, \beta\gamma)$$
$$\alpha \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \beta^t = \alpha\gamma \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \gamma^t \beta^t$$

This means the automorphisms of $(\mathcal{J}(X), \omega_s)$ correspond to matrices, $\gamma \in SL(2g, \mathbb{Z})$ such that:

$$\gamma \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \gamma^t = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

We will call this group of matrices $G_{\mathbb{Z}}$. Then Jacobians of algebraic curves of genus $g$ can be indexed by elements of $H_g/G_{\mathbb{Z}}$, where $H_g$ is the Siegel half plane.

## 6. Riemann's Theorem

Since $(\mathcal{J}(X), \omega_s)$ is a polarized variety, $[\omega_s]$ is necessarily the Chern class of some positive line bundle, $L$, on $\mathcal{J}(X)$ defined up to translation by an element of $\mathcal{J}(X)$. $\omega_s$ corresponds to the intersection form for $H_1(X, \mathbb{Z})$, which has a determinate of $\pm 1$. It can be shown that this implies $H^0(\mathcal{J}(X), L) = 1$. So, up to linear equivalence and translation, $L$ corresponds to a divisor, $\Theta$, on $\mathcal{J}(X)$. This divisor is called the Riemann Theta Divisor.

Given an algebraic curve, $X$, and a basis $\{\omega_1, \ldots, \omega_g\}$ for $H^{1,0}(X, \mathbb{C})$, there is a natural map from $X$ to the corresponding Jacobian:

$$\Phi : X \to \mathcal{J}(X)$$

$$x \mapsto \left(\int_{x_0}^{x} \omega_1, \ldots, \int_{x_0}^{x} \omega_g\right)$$

where $x_0 \in X$ is an arbitrary base point.

Let $S^k X$ be the $k$th symmetric power of $X$. It is the same as the classifying space of sets of $k$ points in $X$, or effective divisors of degree $k$. We can extend $\Phi$ to $S^k X$ as follows:

$$\Phi^{(k)} : S^k X \to \mathcal{J}(X)$$

$$(x_1, \ldots, x_k) \mapsto \sum_{i=1}^{k} \Phi(x_i)$$

Let $W_k = Im\, \Phi^{(k)}$. This is a $k$ dimensional complex surface in $\mathcal{J}(X)$. In particular, $W_{g-1}$ may be regarded as a divisor of $\mathcal{J}(X)$.

The following theorem is due to Abel:

**Theorem 1** (Abel)**.** *$\sum_{i=1}^{k}(x_i)$ is linearly equivalent to $\sum_{i=1}^{k}(y_i)$ as a divisor on $X$ if and only if $\Phi^{(k)}(x_1, \ldots, x_k) = \Phi^{(k)}(x_1, \ldots, x_k)$.*

This means $W_k$ parameterizes equivalence classes of effective divisors on $X$ of degree $k$. The following theorem, which we will not prove either, is due to Riemann.

**Theorem 2** (Riemann)**.** *$\Theta$ and $W_{g-1}$ coincide; i.e. $W_{g-1}$ is a representative of the Riemann Theta class of divisors.*

Notice that $W_{g-1}$ depends on the basis $\{\omega_1, \dots, \omega_g\}$ for $H^{1,0}(X, \mathbb{C})$, while $\Theta$ depends only on the polarized variety, $(\mathcal{J}(X), \omega_s)$.

## 7. Torelli's Theorem

In this paper we are mainly interested in sketching a proof of the following theorem.

**Theorem 3** (Torelli)**.** *If $X_1$ and $X_2$ are two non-singular projective algebraic curves of genus $g$ and $(\mathcal{J}(X_1), \omega_s) \simeq (\mathcal{J}(X_2), \omega_s)$, then $X_1 \simeq X_2$.*

This theorem tells us that an algebraic curve can be recovered from its Jacobian. We will show this by showing that $X$ can be recovered from the divisor $W_{g-1}$ on $\mathcal{J}(X)$. We will consider only the case when $X$ is non-hyperelliptic.

Let $X$ be a non-hyperelliptic non-singular projective algebraic curve. $\mathbb{P}(T_{\mathcal{J}(X)})^*$ is the dual vector bundle to $\mathbb{P}(T_{\mathcal{J}(X)})$ and may be considered to be the bundle of $g-1$ dimensional complex subspaces of $T_{\mathcal{J}(X)}$. There is a natural map from the non-singular points of $W_{g-1}$ into $\mathbb{P}(T_{\mathcal{J}(X)})^*$; namely:

$$w \mapsto \mathbb{P}(T_{W_{g-1},w})$$

Since $\mathcal{J}(X)$ is a torus, $T_{\mathcal{J}(X)}$ is a trivial bundle and we may consider the above map as:

$$i_W : reg(W_{g-1}) \to (\mathbb{P}^{g-1})^* \simeq \mathbb{P}^{g-1}$$

Remember $W_{g-1}$ is the image of $\Phi^{(g-1)}$ in $\mathcal{J}(X)$. Thus to calculate $i_W$ at a point, we write:

$$\Phi^{(g-1)}(x_1, \dots, x_{g-1}) = \Big(\sum \int_{x_0}^{x_i} \omega_1, \dots, \sum \int_{x_0}^{x_i} \omega_1\Big)$$

$$\frac{\partial \Phi^{(g-1)}}{\partial x_i} = \Big(\frac{\omega_1(x_i)}{dx_i}, \dots, \frac{\omega_g(x_i)}{dx_i}\Big)$$

But $(\frac{\omega_1(x_i)}{dx_i}, \dots, \frac{\omega_g(x_i)}{dx_i}) = i(x_i) \in \mathbb{P}^{g-1}$, where $i : X \to \mathbb{P}^{g-1}$ is the canonical map generated by the complete linear system, $|K_X|$. For $X$ non-hyperelliptic this map is an embedding.

We thus have:

$$i_W \circ \Phi^{(g-1)}(x_1, \dots, x_{g-1}) = Span\{i(x_i)\}$$

Let $B$ be the branch locus of $i_W$. That is, $B$ is the image in $(\mathbb{P}^{g-1})^*$ of points, $w \in reg(W_{g-1})$, with $di_W$ degenerate at $w$.

For a curve $Y \subseteq \mathbb{P}^N$, the projective dual, $Y^* \subseteq (\mathbb{P}^N)^*$, is the set of hyperplanes in $\mathbb{P}^N$ tangent to $Y$ at one or more points. We will show $\overline{B} = i(X)^*$.

Given a hyperplane, $H \in (\mathbb{P}^{g-1})^*$, it will intersect $i(X) \subseteq \mathbb{P}^{g-1}$, counting multiplicity, at $deg|K_X| = 2g-2$ points. Let $U = \{H \in (\mathbb{P}^{g-1})^* | \forall \{i(x_1), \ldots, i(x_{g-1})\} \subseteq H \cap i(X),\ \Phi^{(g-1)}(x_1, \ldots, x_{g-1}) \in reg(W_{g-1})\}$. It can be shown that U will be open.

Let $B_1 = \{H \in U | H$ intersects $i(X)$ transversely everywhere but one point where it intersects with multiplicity $2\}$. Any hyperplane which is tangent to $i(X)$ at more than one point will be in the closure of $B_1$. In particular, $i(X)^* \cap U \subseteq \overline{B_1}$.

Let $H \in B_1$ meet $i(X)$ at $i(x_1)$ with multiplicity 2, and let $i(x_2), \ldots, i(x_{g-1})$ be $g-2$ other distinct points in $H \cap i(X)$. Since $H \in B_1 \subseteq U$ we have, $\Phi^{(g-1)}(x_1, \ldots, x_{g-1}) = w \in reg(W_{g-1})$. Notice $i_W(w) = Span\{i(x_i)\} = H$.

Let $z_i$ be a local coordinate for $X$ at $x_i$. Then $(z_1, \ldots, z_{g-1})$ corresponds to local coordinates on $W_{g-1}$ at $w$. Since $H = i_W(w)$ is tangent to $i(X)$ at $i(x_1)$, we may say that up to a first degree variation of $x_1$, $i_W \circ \Phi^{(g-1)}(x_1, \ldots, x_{g-1})$ remains equal to $H$. More precisely that is, $\frac{\partial}{\partial z_1} i_W(w) = 0$. This means $di_W$ is degenerate at $w$. So $H \in B \implies B_1 \subseteq B$.

By the same argument we see that if $H$ intersects $i(X)$ transversely at $i(x_1), \ldots, i(x_{g-1})$, then $di_W$ is non-degenerate at $w = \Phi^{(g-1)}(x_1, \ldots, x_{g-1})$. So if $H$ intersects $i(X)$ transversely at all $2g-2$ points of intersection, it follows that $H \notin B$. This means $H \notin i(X)^* \implies H \notin B$. Thus $B \subseteq i(X)^*$.

But $i(X)^* \cap U \subseteq \overline{B_1} \subseteq \overline{B}$. From the irreducibility of $i(X)^*$ it follows that $i(X) = B$.

Now $B$ is uniquely determined by $W_{g-1}$, which is equivalent to the Riemann Theta Divisor, $\Theta$, which is uniquely determined by the polarized Jacobian, $(\mathcal{J}(X), \omega_s)$. Also $i(X)^*$ uniquely determines $i(X) \subseteq \mathbb{P}^{g-1}$, which determines $X$, since $i$ is an embedding. Thus we see that $X$, up to isomorphism, can be recovered from $(\mathcal{J}(X), \omega_s)$. This is Torelli's theorem for algebraic curves.

## 8. Conclusion

Let $\mathcal{M}_g$ be the classifying space of algebraic curves of genus $g$ up to isomorphism. We have seen that given such a curve, $X$, and bases for $H^1(X, \mathbb{Z})$ and $H^{1,0}(X, \mathbb{C})$, there is a corresponding $g \times g$ period matrix, $Z$, with $Z \in H_g$, since $H^1(X, \mathbb{C})$ has a polarized Hodge structure.

Furthermore, $X$ corresponds to its polarized Jacobian, $(\mathcal{J}(X), \omega_s)$, which is a polarized complex torus with the same period matrix, $Z$. Thus $(\mathcal{J}(X), \omega_s)$ is in the space of polarized complex tori with period matrix in the Siegel half plane. When isomorphisms are quotiented out, this space is classified by $H_g/G_{\mathbb{Z}}$.

Thus the map of isomorphism classes, $X \mapsto (\mathcal{J}(X), \omega_s)$ is a map from $\mathcal{M}_g$ to $H_g/G_{\mathbb{Z}}$. Torelli's theorem for curves states that this map is injective. We therefore have:

$$\mathcal{M}_g \hookrightarrow H_g/G_{\mathbb{Z}}$$

which is a nice result to draw from Torelli's theorem.