$\boxed{1}$ (10 points) Consider the integral:

$$\int_0^4 \int_{y^2/4}^4 y\cos(x^2)\ dx\ dy$$

(a) Sketch the region in the x, y plane corresponding to this integral.

(b) Reverse the order of integration by treating the region as the other type.

(c) Compute the integral.

Solution:

(a) Since $x = y^2/4$ meets $x = 4$ at $(4, 4)$, the region corresponding to the integral is the region in the x, y plane in the first quadrant bounded by the curves $x = y^2/4$, $x = 4$ and $y = 0$.

(b) The integral as written is type II. To write it as type I we want to write $x = y^2/4$ with y being a function of x; that is,

$$y^2 = 4x$$
$$y = 2\sqrt{x}$$

Then the integral as type I becomes,

$$\int_0^4 \int_0^{2\sqrt{x}} y\cos(x^2)\ dy\ dx$$

(c) This integral is now possible to evaluate,

$$\begin{aligned}\int_0^4 \int_0^{2\sqrt{x}} y\cos(x^2)\ dy\ dx &= \int_0^4 \frac{1}{2}y^2\cos(x^2)\Big|_0^{2\sqrt{x}} dx\\ &= \int_0^4 2x\cos(x^2)dx\\ &= \sin(x^2)\Big|_0^4 = \sin 16\end{aligned}$$

$\boxed{2}$ (10 points) In polar coordinates $r = \theta,\ \theta \geq 0$ represents a spiral. The first "revolution" of this spiral can be parameterized as, $x(t) = t\cos t,\ y(t) = t\sin t,\ 0 \leq t \leq 2\pi$. Call this portion of the spiral, C.

Find,

$$\int_C \sqrt{x^2+y^2}\ ds$$

Solution: Our parameterization for the curve C is,

$$\mathbf{r}(t) = \langle t\cos t, t\sin t\rangle,\ 0 \leq t \leq 2\pi$$

To compute the line integral we need to write the correct limits, rewrite the function in terms of t and rewrite ds in terms of t and dt. We have,

$$\sqrt{x^2+y^2} = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = t$$

$$ds = \sqrt{x'(t)^2 + y'(t)^2}dt$$

with,

$$x'(t) = \cos t - t\sin t$$
$$y'(t) = \sin t + t\cos t$$

then,

$$\begin{aligned} ds &= \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}dt \\ &= \sqrt{1+t^2}dt \end{aligned}$$

Thus,

$$\int_C \sqrt{x^2+y^2}ds = \int_0^{2\pi} t\sqrt{1+t^2}dt$$

making a substitution, $u = 1+t^2$, allows the integral to be evaluated as,

$$\frac{1}{3}(1+t^2)^{3/2}\Big|_0^{2\pi} = \frac{1}{3}\left[(1+4\pi^2)^{3/2} - 1\right]$$

$\boxed{3}$ (10 points) Suppose the shape of a small open cone is described by the surface,

$$S: \ z = 2\sqrt{x^2+y^2}, \ 0 \le z \le 4,$$

where units of length are cm.s.
If the cone has constant density ρ g./cm.^{2}, find its moment of inertia about the z-axis, I_z.
(Hint: You may want to parameterize S in terms of r and θ)

Solution:
The equation for I_z is,

$$I_z = \iint_S (x^2+y^2)\rho dS$$

where (x^2+y^2) is the distance from the z-axis.
To calculate this surface integral, we must first parameterize the surface and then write the limits of integration, the function and the infintesimal dS in terms of the parameters.
If we parameterize in terms of r and θ, we will have, $x = r\cos\theta$, $y = \sin\theta$. This means $z = 2r$. Since we have $0 \le z \le 4$, we will have $0 \le r \le 2$. Thus the parameterization may be written as,

$$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 2r\rangle, \ 0 \le \theta \le 2\pi, \ 0 \le r \le 2$$

We have,

$$(x^2+y^2) = (r^2\cos^2\theta + r^2\sin^2\theta) = r^2$$

For infintesimal surface area, we have, $dS = |\mathbf{r}_r \times \mathbf{r}_\theta| dr d\theta$. We may calculate,

$$\mathbf{r}_r = \langle \cos\theta, \sin\theta, 2\rangle$$

$$\mathbf{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$$

$$\mathbf{r}_r \times \mathbf{r}_\theta = \langle -2r\cos\theta, -2r\sin\theta, r\rangle$$

$$|\mathbf{r}_r \times \mathbf{r}_\theta| = r\sqrt{5}$$

Thus we have,

$$\begin{aligned} I_z &= \int_0^{2\pi}\int_0^2 r^2\rho r\sqrt{5}drd\theta \\ &= \rho\sqrt{5}\int_0^{2\pi} \frac{1}{4}r^4\Big|_0^2 d\theta \\ &= \rho\sqrt{5}\int_0^{2\pi} 4d\theta = 8\pi\sqrt{5}\rho \end{aligned}$$

$\boxed{4}$ (10 points) Let R be the region in the first quadrant of the x, y plane bounded by the curves, $xy = 1,\ y = 4x,\ y = x/4$.
Consider the change of variables, $x = u/v,\ y = uv,\ u, v \geq 0$.

(a) Sketch the region R in the x, y plane and the corresponding region D in the u, v plane. (notice that the line $u = 0$ corresponds to $(0, 0)$ in the x, y plane.)

(b) Find the Jacobian, $\frac{\partial(x,y)}{\partial(u,v)}$.

(c) Use the change of variables to find the area of R by evaluating $\iint_R 1\ dxdy$.

Solution:

(a) One way to find D is to find the curves in the u, v plane that correspond to the boundary curves of R. Notice that we are restricting both u and v to be non-negative. This corresponds to x and y also being non-negative.

$$xy = 1 \quad \longrightarrow \quad u^2 = 1 \quad \longrightarrow \quad u = 1$$

$$y = 4x \quad \longrightarrow \quad uv = 4u/v \quad \longrightarrow \quad v^2 = 4 \quad \longrightarrow \quad v = 2$$

$$y = x/4 \quad \longrightarrow \quad uv = u/4v \quad \longrightarrow \quad v^2 = 1/4 \quad \longrightarrow \quad v = 1/2$$

We may also notice, as in the hint, that the origin in the x, y plane corresponds to $u = 0$. Thus the corresponding region, D in the u, v plane is the rectangle,

$$D = \{(u, v)|\ 0 \leq u \leq 1, \frac{1}{2} \leq v \leq 2\}$$

(b)

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} x_u & y_u \\ y_u & y_v \end{vmatrix} \\
&= \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} \\
&= 2\frac{u}{v}
\end{aligned}$$

(c)

$$\begin{aligned}
\text{Area}(R) = \iint_R dxdy &= \iint_D \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \\
&= \int_{1/2}^{2} \int_0^1 2\frac{u}{v} dudv \\
&= \int_{1/2}^{2} \frac{1}{v} dv \\
&= \ln|v| \Big|_{1/2}^{2} = 2\ln 2
\end{aligned}$$