1 (8 points) Consider the parameterized curve in space:

$$\mathbf{r}(t) = \langle 2\cos t, 3\sin t, t\rangle, \ -\infty < t < \infty$$

Let $f(x, y, z)$ be a smooth function defined everywhere with:

$$\nabla f = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$$

Write an expression for: $\frac{d}{dt}f(\mathbf{r}(t))$ as a function of t. (This will involve the functions P, Q, and R)

Solution:

$$\frac{d}{dt}f(\mathbf{r}(t)) = \nabla f \cdot \mathbf{r}'(t)$$

$$= \langle P, Q, R\rangle \cdot \langle -2\sin t, 3\cos t, 1\rangle$$

$$= -2(\sin t)P + 3(\cos t)Q + R$$

$$= -2(\sin t)P(2\cos t, 3\sin t, t) + 3(\cos t)Q(2\cos t, 3\sin t, t) + R(2\cos t, 3\sin t, t)$$

2 (12 points) Consider the curve in the plane:

$$\mathbf{r}(t) = \langle t\cos t, t\sin t\rangle,\ -\pi/2 \leq t \leq \pi/2$$

This is a simple closed curve bounding a region D. Use Green's Theorem and the vector field $\mathbf{F} = -\frac{y}{2}\mathbf{i} + \frac{x}{2}\mathbf{j}$ to compute the area of D.

Solution:
According to Green's Theorem:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_D d\mathbf{F}\ dA$$

Now,

$$d\mathbf{F} = \frac{\partial}{\partial x}\left(\frac{x}{2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{2}\right) = 1$$

This means:

$$\text{Area}(D) = \iint d\mathbf{F}\ dA$$

$$= \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{-\pi/2}^{\pi/2} \mathbf{F(r(t))\cdot r'(t)dt}$$

$$= \int_{-\pi/2}^{\pi/2} \langle -\frac{t\sin t}{2}, \frac{t\cos t}{2}\rangle \cdot \langle \cos t - t\sin t, \sin t + t\cos t\rangle dt$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2}t^2 dt = \frac{\pi^3}{24}$$

3 (16 points) This question is short answer. No justification will be necessary and no partial credit will be given for the items.
Let $f(x, y, z) = xyz + \sin z$, $\mathbf{F} = \langle xe^y, \sin z, x + y + z\rangle$, and $\mathbf{u}$ be the unit vector in the same direction as $\langle 3, 4, 12\rangle$.

(a) Find $\operatorname{grad} f$.

(b) Find $\nabla^2 f$.

(c) Find $D_{\mathbf{u}} f$ at (1,2,0).

(d) Find $\nabla \times \mathbf{F}$.

(e) Let C be the straight line path from $(0, 0, 0)$ to $(1, 1, \pi)$. Find $\int_C \nabla f \cdot d\mathbf{r}$.

(f) Let S be the surface of the sphere of radius 2 centered at the origin with outward pointing normal. Find $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

(g) Find $\operatorname{div} \operatorname{curl} \mathbf{F}$.

(h) Let C be the circle $\{(x, y, z)|x^2 + y^2 = 1,\ z = 0\}$, and let E be all of x, y, z space except for C. Is E simply connected?

Solutions:

(a) $\nabla f = \langle yz, xz, xy + \cos z\rangle$

(b) $\nabla^2 f = -\sin z$

(c)

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$
$$\nabla f(1, 2, 0) = \langle 0, 0, 3\rangle$$
$$\mathbf{u} = \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\rangle$$
$$D_{\mathbf{u}} f = \frac{36}{13}$$

(d) $\nabla \times \mathbf{F} = \langle 1 - \cos z, -1, -xe^y\rangle$

(e) By the Fundamental Theorem of Line Integrals:

$$\int_C \nabla f \cdot d\mathbf{r} = f(1, 1, \pi) - f(0, 0, 0) = \pi$$

(f) $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = 0$, since S is a closed surface.

(g) $\operatorname{div} \operatorname{curl}\mathbf{F} = 0$. This is true for all differentiable vector fields.

(h) E is not simply connected. To see this consider the simple closed curve given by:

$$C' = \{(x, y, z)|\ (x - 1)^2 + z^2 = 1,\ y = 0\}$$

Since the circle C is removed from space, the circle C' is contained in E, but is not the boundary of any surface lying entirely in E.

4 (14 points) Let C be the simple closed curve parameterized as:

$$\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 2\sin t\rangle,\ 0 \leq t \leq 2\pi$$

Notice C is the boundary of a surface parameterized as:

$$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\sin\theta\rangle,\ 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi$$

Let $\mathbf{F} = \langle ye^x, e^x + z^2, \sin y + x\rangle$. Use Stoke's Theorem to find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Solution:
Let S be the parameterized surface described. Since C has counter-clockwise orientation when viewed from above, the corresponding orientation on S is an upward pointing normal. This corresponds to:

$$\mathbf{r}_r \times \mathbf{r}_\theta = \langle 0, -r, r\rangle$$

Then according to Stoke's Theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot \mathbf{S}$$

$$= \int_0^{2\pi} \int_0^2 \text{curl}\mathbf{F} \cdot (\mathbf{r}_r \times \mathbf{r}_\theta) dr d\theta$$

We have:

$$\text{curl}\mathbf{F} = \langle \cos y - 2z, -1, 0\rangle$$

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \int_0^2 r\ dr d\theta = 4\pi$$

5 (10 points) Let $E = \{(x, y, z)|\ x^2 + y^2 \leq 1,\ -1 + x^2 + y^2 \leq z \leq 1 - x^2 - y^2\}$. Let S be the boundary surface of E with outward pointing normal. Let $\mathbf{F} = \langle xy^2, x^2y - \sin z, x^3 + z\rangle$. Use the Divergence Theorem to find $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Solution:
According to the Divergence Theorem:

$$\iint_S \mathbf{F} \cdot \mathbf{S} = \iiint_E \text{div}\mathbf{F} dV$$

We have div$\mathbf{F} = y^2 + x^2 + 1$. Also, let $D = \{(x, y)|x^2 + y^2 \leq 1\}$ be the "footprint" of E in the x, y plane. Then we may do the triple integral as a Type I integral.

$$\iiint_E \text{div}\mathbf{F} dV = \iint_D \int_{-1+x^2+y^2}^{1-x^2-y^2} y^2 + x^2 + 1\ dzdA$$

$$= \iint_D 2(1 + x^2 + y^2)(1 - x^2 - y^2)dA$$

$$= \int_0^{2\pi} \int_0^1 2(1 + r^2)(1 - r^2)rdrd\theta = \frac{4\pi}{3}$$