(1) Find the surface area of the surface $z = \frac{1}{2}(x^2 + y^2)$ over the region $D = \{(x, y)|x^2 + y^2 \leq 1, -x \leq y \leq x\}$.

Solution: According to the formula for surface area we have:

$$A = \iint_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} dA$$

We have $\frac{\partial z}{\partial x} = x$, $\frac{\partial z}{\partial y} = y$. So we are integrating the function $\sqrt{1 + x^2 + y^2}$ over D. This will be easiest to do using polar coordinates. We rewrite D as:

$$D = \{(r, \theta)|0 \leq r \leq 1, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\}$$

$\sqrt{1 + x^2 + y^2}$ becomes $\sqrt{1 + r^2}$, and we remember that dA is replaced by $rdrd\theta$. Now we integrate:

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^1 \sqrt{1 + r^2} rdrd\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{3}(1 + r^2)^{\frac{3}{2}}|_0^1 d\theta$$

$$= \frac{1}{3} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2\sqrt{2} - 1) d\theta = \frac{\pi}{6}(2\sqrt{2} - 1)$$

(2) Suppose we wish to make a thin metal washer with constant density, ρ g/cm.^{2}, with outer radius 2 cm., and inner radius a cm. for some constant a, $0 < a < 2$. We represent the washer by the region in the x, y plane, $D = \{(x, y)| \ a^2 \leq x^2 + y^2 \leq 4\}$.

(a) Compute a formula for the mass of the washer, m, in terms of a and ρ.

(b) Compute a formula for the moment of intertia of the washer about the y-axis, I_y, in terms of a and ρ.

(c) If we want $m = 9$ g. and $I_y = 13$ g.cm.^{2}, what value must we choose for a?

Solution:

(a) We know:

$$\text{mass} = \iint_{\text{D}} \rho \text{dA}$$

We can write this as a polar integral as follows:

$$\int_0^{2\pi} \int_a^2 \rho r \ dr \ d\theta$$

Then we compute this integral. Alternatively we can observe that mass is area times density. Therefore:

$$\text{mass} = \text{Area(D)}\rho = (4\pi - \pi \text{a}^2)\rho = \pi\rho(4 - \text{a}^2)$$

(b) We know:

$$I_y = \iint_D x^2 \rho dA$$

We want to write this in polar coordinates since D is then easily described. We replace x with $r \cos\theta$ and dA with $r \ dr \ d\theta$. Then:

$$I_y = \int_0^{2\pi} \int_a^2 r^2 \cos^2\theta \ \rho r \ dr \ d\theta$$

$$= \int_0^{2\pi} \frac{1}{4} r^4 \cos^2\theta \ \rho|_a^2 \ d\theta$$

$$= \frac{\rho}{4}(16 - a^4) \int_0^{2\pi} \cos^2\theta \ d\theta$$

We calculate this integrle by using the half angle formula:

$$\int_0^{2\pi} \cos^2\theta \ d\theta = \int_0^{2\pi} \frac{1}{2}(1 + \cos(2\theta))d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)|_0^{2\pi} = \pi$$

Therefore:

$$I_y = \frac{\pi\rho}{4}(16 - a^4)$$

(c) We have:

$$m = 9 = \pi\rho(4 - a^2)$$

$$I_y = 13 = \frac{\pi\rho}{4}(16 - a^4)$$

Then:

$$\frac{I_y}{m} = \frac{13}{9} = \frac{16 - a^4}{4(4 - a^2)}$$

$$\frac{13}{9} = \frac{(4 - a^2)(4 + a^2)}{4(4 - a^2)} = \frac{4 + a^2}{4}$$

$$4 + a^2 = \frac{52}{9}$$

$$a^2 = \frac{16}{9}$$
$$a = \frac{4}{3}$$

since a is a positive number.

(3) Usa a change of variables to evaluate the double integral

$$\int\int_D e^{2x^2+9y^2}\ dA,$$

where D is the elliptical domain $2x^2 + 9y^2 \leq 1$.

Note: if you forget how to approach this problem, then for a maximum of 7 points you may integrate:

$$\int\int_D e^{x^2+y^2}\ dA,$$

where D is the domain $x^2 + y^2 \leq 1$.

Solution:
Use change of coordinates, $u = \sqrt{2}x,\ v = 3y$ Then:

$$\iint_D e^{2x^2+9y^2} dA = \frac{\pi\sqrt{2}(e-1)}{6}$$

(4) Let S denote the surface defined by the equation $x^2 + y^2 + z = 4$ above the xy-plane.
  (a) Find a parametrization for S.
  (b) Evaluate:

$$\int\int_S \frac{1}{\sqrt{1 + 4x^2 + 4y^2}} dS.$$

Solution:

(a)

$$\mathbf{r}(r, \theta) = \left(r\cos\theta, r\sin\theta, 4 - r^2\right), \; 0 \leq \theta \leq 2\pi, \; 0 \leq r \leq 2$$

(b)

$$\int_0^{2\pi} \int_0^2 \frac{1}{\sqrt{1 + 4r^2}} |\mathbf{r}_r \times \mathbf{r}_\theta| dr d\theta = 4\pi$$

(5) The hyperbolic sine and cosine functions are given by:

$$\cosh t = \frac{e^t + e^{-t}}{2}, \qquad \sinh t = \frac{e^t - e^{-t}}{2}$$

(Notice $\cosh^2 t - \sinh^2 t = 1$) The object of this problem is to use a change of variables, $x = r\cosh t$, $y = r\sinh t$, to compute $\iint_D x - y\; dA$, where $D = \{(x,y) |\ 0 \leq y, y \leq x, x^2 - y^2 \leq 1\}$.

(a) Compute the Jacobian $\frac{\partial(x,y)}{\partial(r,t)}$

(b) Describe the region that corresponds to D in the r, t plane.

(c) Use change of variables to compute $\iint_D x - y\; dA$.

Solution:

(a) By definition:

$$\frac{\partial(x,y)}{\partial(r,t)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial t} \end{vmatrix}$$

Now $\frac{\partial x}{\partial r} = \cosh t$ and $\frac{\partial y}{\partial r} = \sinh t$. Notice:

$$\frac{d}{dt}\cosh t = \frac{d}{dt}\left(\frac{e^t + e^{-t}}{2}\right) = \frac{e^t - e^{-t}}{2} = \sinh t$$

Similarly $\frac{d}{dt}\sinh t = \cosh t$. Therefore, $\frac{\partial x}{\partial t} = r\sinh t$ and $\frac{\partial y}{\partial t} = r\cosh t$. Then:

$$\frac{\partial(x,y)}{\partial(r,t)} = \begin{vmatrix} \cosh t & r\sinh t \\ \sinh t & r\cosh t \end{vmatrix} = r\cosh^2 t - r\sinh^2 t = r(\cosh^2 t - \sinh^2 t) = r$$

(b) We will replace $y \leq x$ with $y < x$. This will not affect the final integral, since it removes from the region a ray of no area. This is a technicality and due to an oversight of mine in writing the problem.

Since $0 \leq y$, $y < x$ is the same as $y^2 < x^2$ or $0 < x^2 - y^2$. Thus we have $0 \leq y$ and $0 < x^2 - y^2 \leq 1$. But:

$$x^2 - y^2 = r^2\cosh^2 t - r^2\sinh^2 t = r^2$$

So we have $0 < r^2 \leq 1$. Notice $0 < \cosh t$ for all t. Therefore $0 < x = r\cosh t$ tells us that $0 < r$. So we know $0 < r \leq 1$ and $0 \leq y$. Now:

$$0 \leq y = r\sinh t \implies 0 \leq \sinh t = \frac{e^t - e^{-t}}{2}$$

$$\implies e^{-t} \leq e^t \implies 0 \leq t$$

Thus the corresponding region in the r, t plane is the unbounded "rectangle":

$$R = \{(r,t) |\ 0 < r \leq 1,\ 0 \leq t < \infty\}$$

(c) Change of variables tells us:

$$\iint_D x - y\; dA = \iint_R x(r,t) - y(r,t)\left|\frac{\partial(x,y)}{\partial(r,t)}\right|\; dA$$

$$= \int_0^\infty \int_0^1 (r\cosh t - r\sinh t) r\; dr\; dt = \int_0^\infty \int_0^1 r^2 e^{-t}\; dr\; dt$$

$$= \int_0^\infty \frac{1}{3} r^3 e^{-t}\Big|_0^1\; dt$$

$$= \frac{1}{3}\int_0^\infty e^{-t}\; dt = \frac{1}{3}$$