Your Name

Student ID #

Problem	Total Points	Score
1	14	
2	14	
3	14	
4	8	
Total	50	

- This exam is closed book. You may have one sheet of handwritten notes and a non-graphing calculator.
- In order to receive credit, you must show your work. You must also justify all conclusions you make. Do not do computations in your head. Instead, write them out on the exam paper.
- Place a box around $\boxed{\text{YOUR FINAL ANSWER}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- GOOD LUCK!

$\boxed{1}$ (14 points) Consider the integral:

$$\int_0^1 \int_0^{1-x} \frac{1}{\sqrt{xy}}\, dy\, dx$$

(a) Sketch the region in the x, y plane corresponding to this integral.

(b) Use a change of variables, $u = \sqrt{x},\ v = \sqrt{y}$, to compute the integral.

Solution:

(a) The region integrated over is:

$$R = \{(x, y)|\ 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x\}$$

This is the right triangular region bounded by $x = 0$, $y = 0$ and $y = 1 - x$.

(b) Our change of variables is given by $x = u^2$ and $y = v^2$. First we will find the region corresponding to R in the u, v plane. We have:

$$0 \leq x \leq 1 \quad 0 \leq y \leq 1 - x$$

$$0 \leq u^2 \leq 1 \quad 0 \leq v^2 \leq 1 - u^2$$

Since u and v are non-negative, this means the region in the u, v plane is:

$$D = \{(u, v)|\ 0 \leq u \leq 1,\ 0 \leq v \leq \sqrt{1 - u^2}\}$$

This is the portion of the unit disc in the first quadrant.
By the change of variable formula we know:

$$\int_0^1 \int_0^{1-x} \frac{1}{\sqrt{xy}}\, dy\, dx = \iint_R \frac{1}{\sqrt{xy}}\, dy\, dx = \iint_D \frac{1}{uv} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Thus we must compute the Jacobian:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 0 \\ 0 & 2v \end{vmatrix} = 4uv$$

$$\iint_R \frac{1}{\sqrt{xy}}\, dy\, dx = \iint_D 4\, du\, dv = 4\text{Area(D)} = \pi$$

This is because D is a quarter of a unit disc and so has area $\pi/4$.

$\boxed{2}$ (14 points) For constants, $0 < R_2 < R_1$, consider the surface, T, described by the following parameterization:

$$\mathbf{r}(u,v) = \langle (R_1 + R_2\cos v)\cos u,\ (R_1 + R_2\cos v)\sin u,\ R_2\sin v\rangle$$

$$0 \le u \le 2\pi, \quad 0 \le v \le 2\pi$$

This surface resembles the surface of a doughnut; it is an example of a torus.
Find the surface area of T; this will depend on R_1 and R_2.
(Remember: surface area $= \iint_T 1\ dS$)

Solution: This is a somewhat long computation based on:

$$\iint_T dS = \int_0^{2\pi}\int_0^{2\pi} |\mathbf{r}_u \times \mathbf{r}_v|\ du\ dv$$

As follows:

$$\mathbf{r}_u = \langle -(R_1 + R_2\cos v)\sin u, (R_1 + R_2\cos v)\cos u, 0\rangle$$

$$\mathbf{r}_v = \langle -R_2\sin v\cos u, -R_2\sin v\sin u, R_2\cos v\rangle$$

The next two lines require some use of the identities: $\sin^2 u + \cos^2 u = \sin^2 v + \cos^2 v = 1$.

$$\mathbf{r}_u \times \mathbf{r}_v = \langle R_2(R_1 + R_2\cos v)\cos u\cos v, R_2(R_1 + R_2\cos v)\sin u\cos v, R_2(R_1 + R_2\cos v)\sin v\rangle$$

$$|\mathbf{r}_u \times \mathbf{r}_v| = R_2(R_1 + R_2\cos v)$$

Finally,

$$\iint_T dS = \int_0^{2\pi}\int_0^{2\pi} R_2(R_1 + R_2\cos v)\ du\ dv = 4\pi^2 R_1 R_2$$

$\boxed{3}$ (14 points) A wire of variable density is circular, described by the curve $x^2 + y^2 = 1$. Suppose the density is given by $\rho(x, y) = 2 + x$.
Find the center of mass of the wire.
(note: you can parameterize the curve as $\mathbf{r}(t) = \langle \cos t, \sin t \rangle,\ 0 \le t \le 2\pi$)

Solution: To find the center of mass of the wire we must find its mass, m, and its moments about the x and y axes, M_x and M_y. To start, call the curve we are interested C. Then the mass of the wire is given by a line integral of ρ along C with respect to arclength:

$$m = \int_C \rho(x, y) ds = \int_0^{2\pi} (2 + x(t)) \sqrt{x'(t)^2 + y'(t)^2}\ dt$$

Now,

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$$

Thus

$$m = \int_0^{2\pi} 2 + \cos t\ dt = 4\pi$$

$$M_x = \int_C \rho(x, y) y\ ds = \int_0^{2\pi} (2 + x(t)) y(t) \sqrt{x'(t)^2 + y'(t)^2}\ dt$$

$$= \int_0^{2\pi} (2 + \cos t) \sin t\ dt = 0$$

$$M_y = \int_C \rho(x, y) x\ ds = \int_0^{2\pi} (2 + x(t)) x(t) \sqrt{x'(t)^2 + y'(t)^2}\ dt$$

$$= \int_0^{2\pi} (2 + \cos t) \cos t\ dt = \pi$$

We can now compute $\bar{x}$ and $\bar{y}$.

$$\bar{x} = \frac{M_y}{m} = \frac{\pi}{4\pi} = \frac{1}{4} \quad \bar{y} = \frac{M_x}{m} = 0$$

Thus the center of mass is at the point $(1/4, 0)$.

$\boxed{4}$ (8 points) Consider the surface: $z = xy,\ x^2 + y^2 \leq 9$
Call this surface S.

(a) Find a parameterization of S in terms of parameters u and v.

(b) Find two non parallel vectors tangent to S at the point $P = (2, 1, 2)$.

(c) Find a parameterization of the plane tangent to S at P in terms of parameters s and t.

Soltuion:

(a) The surface is most easily parameterized by $x = u$, $y = v$, $z = uv$. That is:

$$\mathbf{r}(u, v) = \langle u, v, uv\rangle \quad u^2 + v^2 \leq 9$$

(b) Given a parameterization, $\mathbf{r}_u$ and $\mathbf{r}_v$ are always tangent to the surface. In our case:

$$\mathbf{r}_u = \langle 1, 0, v\rangle \quad \mathbf{r}_v = \langle 0, 1, u\rangle$$

We want to find these vectors at the point $(2, 1, 2)$ on the surface. At the is point we have $u = x = 2$ and $v = y = 1$. Thus the corresponding tangent vectors are:

$$\mathbf{r}_u(2, 1) = \langle 1, 0, 1\rangle \quad \mathbf{r}_v(2, 1) = \langle 0, 1, 2\rangle$$

(c) We want to parameterize a plane containing the point $(2, 1, 2)$ and having tangent vectors, $\langle 1, 0, 1\rangle$ and $\langle 0, 1, 2\rangle$. We write:

$$\mathbf{r}(s, t) = \langle 2, 1, 2\rangle + s\langle 1, 0, 1\rangle + t\langle 0, 1, 2\rangle$$

$$= \langle 2 + s, 1 + t, 2 + s + 2t\rangle$$