1. (10 points) For this problem units of length will be in centimeters.
Consider the segment of the curve $y = x^2$ between $(0, 0)$ and $(\sqrt{3}/2, 3/4)$. Suppose the shape of a piece of wire is described by this curve segment. Furthermore suppose the wire has varying thickness, so that its density is given by $\rho = 2xy$ g./cm.
Use a line integral with respect to arc length to compute the mass of the wire.

Solution:
The shape of the wire is described by a curve, C, in the x, y plane. Then the mass is calculated by integrating $\rho(x, y)$ over C with respect to arc length. That is:

$$\text{mass} = \int_C \rho(x, y)\; ds = \int_C 2xy\; ds$$

To compute this we must parameterize C. This can be done in many ways. An easy way is to let $x = t$, then $y = t^2$, so C is parameterized as,

$$\mathbf{r}(t) = \langle t, t^2 \rangle,\; 0 \le t \le \sqrt{3}/2$$

Another way would be to let $y = t$, $x = \sqrt{t}$ and have $0 \le t \le 3/4$. We will use the former parameterization for this solution.
Then the integral becomes,

$$\begin{aligned} m = \int_C 2xy\; ds &= \int_0^{\sqrt{3}/2} 2x(t)y(t)\sqrt{x'(t)^2 + y'(t)^2}dt \\ &= \int_0^{\sqrt{3}/2} 2(t)(t^2)\sqrt{1^2 + (2t)^2}dt \\ &= \int_0^{\sqrt{3}/2} 2t^3\sqrt{1 + 4t^2}dt \end{aligned}$$

We can perform a change of variable: $u = 1 + 4t^2$. Then $du = 8t\; dt$ and $t^2 = \frac{1}{4}(u - 1)$. When $t = 0$, $u = 1$; when $t = \sqrt{3}/2$, $u = 4$. Then,

$$m = \frac{1}{16}\int_1^4 (u - 1)\sqrt{u}du = \frac{1}{16}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^4 = \frac{29}{60}$$