Toward Arakelov-Parshin Rigidity

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Fixed Notation.

- B is a smooth (not necessarily projective) curve.
- $f : X \to B$ is a smooth projective family of varieties (of general type) of dimension n.
- $X_b = f^{-1}(b)$ is the fiber of f over $b \in B$.

Definition. f is called *isotrivial* if $X_a \simeq X_b$ for $a, b \in B$ general points.

Kodaira-Spencer map.

$$0 \to T_{X/B} \to T_X \to f^*T_B \to 0$$

induces

$$\rho_f : T_B \to R^1 f_* T_{X/B},$$

the Kodaira-Spencer map of f.

Fact.

f is isotrivial $\Leftrightarrow \rho_f = 0$, i.e.,

f is non-isotrivial $\Leftrightarrow \rho_f \neq 0 \Leftrightarrow \rho_f$ is injective.

First assume that $n = 1$, i.e., f is a family of curves.

Definition. Let $g \geq 2$ be fixed. Non-isotrivial families of curves of genus g will be called *admissible*.

Shafarevich's Conjecture. (**SC**)
For a fixed g, there exist only finitely many isomorphism classes of admissible families.

The Arakelov-Parshin method.

Boundedness (**B**): There exist only finitely many deformation types of admissible families.

Rigidity (**R**): There exist no non-trivial deformations of admissible families.

Observe, that (**B**) and (**R**) together imply (**SC**).

A word about the proof.

Let $\bar{B}$ be the projective closure of B and $\bar{f} : \bar{X} \to \bar{B}$ a flat extension of f with $\bar{X}$ smooth and projective.

To prove (**B**), one proves that $\deg(\bar{f}_* \omega^m_{\bar{X}/\bar{B}})$ is bounded in terms of fixed numerical invariants for $m \gg 0$.

To prove (**R**), one proves that if f is non-isotrivial, then $\omega_{\bar{X}/\bar{B}}$ is ample on $\bar{X}$.

By Kodaira vanishing $\omega_{\bar{X}/\bar{B}}$ ample implies that

$$H^1(\bar{X}, T_{\bar{X}/\bar{B}}) = H^1(\bar{X}, \omega^{-1}_{\bar{X}/\bar{B}}) = 0.$$

In other words, f has no first order deformations, and hence (**R**) holds.

Remark. This proof only works in the case of families of curves, i.e., when $\dim X - \dim B = 1$.

<u>GOAL:</u> Generalize (**B**) and (**R**) for higher dimensional families.

- B is a smooth (not necessarily projective) curve.

- $f : X \to B$ is a smooth projective family of varieties of general type of dimension n. (n is arbitrary).

Definition. Fix a polynomial h. Non-isotrivial families of smooth projective varieties of general type with Hilbert polynomial h will be called *admissible*.

<u>Boundedness</u> (**B**): Admissible families are parametrized by a scheme of finite type.

<u>Rigidity</u> (**R**): There exist no non-trivial deformations of admissible families.

Boundedness – Holds.

Theorem. (Bedulev-Viehweg) Let $\bar{B}$ be the projective closure of B and $\bar{f} : \bar{X} \to \bar{B}$ an extension of f. Then for $m \gg 0$, there exists a constant c depending only on m, n, $K^n_{X_b}$, $g(\bar{B})$, $\#(\bar{B} \setminus B)$, such that

$$\deg\left(\bar{f}_*\omega^m_{\bar{X}/\bar{B}}\right) \leq c \cdot \mathrm{rk}\left(\bar{f}_*\omega^m_{\bar{X}/\bar{B}}\right)$$

Remark. Generalizations and related work by Oguiso-Viehweg, Viehweg-Zuo and K____.
(cf. Zuo's lecture)

Rigidity – Fails.

Example. Let $Y \to B$ be a non-isotrivial family of smooth projective curves, and F an arbitrary smooth projective curve. Let $f : X = Y \times F \to B$. Deforming F gives a deformation of f

Problem. Under what additional condition does (**R**) hold?

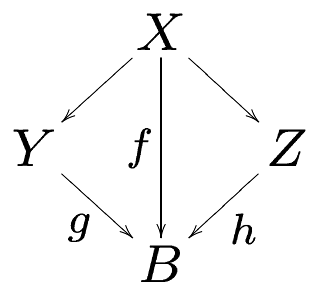
Notes.

- Let $B_0 \subseteq B$ be open and $X_0 = f^{-1}(B_0)$.
  A sufficient condition, if it exists, should be independent of B.
  If it holds for f, it should also hold for $f|_{X_0}$.

- If (**R**) fails for f, it also fails for $f|_{X_0}$.

Principle. When studying (**R**), we will freely restrict to open subsets of B.

Example. Let $g : Y \to B$ and $h : Z \to B$ be two families of smooth projective curves of genus at least two. Let $X = Y \times_B Z$.



- If either g or h is isotrivial, then by the above principle we may assume that it is actually trivial.

  Hence (**R**) fails for f.

- It is easy to prove that

  $$H^1(X, T_{X/B}) \simeq H^1(Y, T_{Y/B}) \oplus H^1(Z, T_{Z/B}).$$

  Recall that if g and h are both non-isotrivial, then $H^1(Y, T_{Y/B}) = H^1(Z, T_{Z/B}) = 0$.

  Hence (**R**) holds for f.

Simple Question. Let $g : Y \to B$ and $h : X \to Y$ be non-isotrivial families of curves of genus at least two.

$$X \xrightarrow{h} Y \xrightarrow{g} B, \qquad f = g \circ h : X \to B$$

Does (**R**) hold for f?

Expectation. YES.

Answer. YES.

Reason. Later.

Back to the product.

$$
\begin{array}{c}
X = Y \times_B Z \\
\Big\downarrow f \\
B
\end{array}
\quad = \quad
\begin{array}{c}
Y \\
\Big\downarrow g \\
B
\end{array}
\times
\begin{array}{c}
Z \\
h \Big\downarrow \\
B
\end{array}
$$

Kodaira-Spencer maps.

$$\rho_g : T_B \to R^1 g_* T_{Y/B}$$

$$\rho_h : T_B \to R^1 h_* T_{Z/B}$$

$$\rightsquigarrow$$

$$\boxed{\rho_g \otimes \rho_h : T_B^{\otimes 2} \to R^1 g_* T_{Y/B} \otimes R^1 h_* T_{Z/B}}$$

Corollary. $\rho_g \otimes \rho_h \neq 0$ implies that (**R**) holds for f.

Notation. $\wedge^m T_X$ will be denoted by T_X^m.

Observe. By the Künneth formula,

$$R^1 g_* T_{Y/B} \otimes R^1 h_* T_{Z/B} \simeq R^2 f_* T^2_{X/B}.$$

Iterated Kodaira-Spencer maps.

$$0 \to T_{X/B} \to T_X \to f^* T_B \to 0 \qquad \rightsquigarrow$$

$$0 \to T^2_{X/B} \to T^2_X \to T_{X/B} \otimes f^* T_B \to 0 \qquad \rightsquigarrow$$

$$\boxed{\rho_f^{(2)} : R^1 f_* T_{X/B} \otimes T_B \to R^2 f_* T^2_{X/B}}$$

$$0 \to T_{X/B} \otimes f^* T_B \to T_X \otimes f^* T_B \to f^* T_B^{\otimes 2} \to 0$$

$$\boxed{\rho_f^{(1)} : T_B^{\otimes 2} \to R^1 f_* T_{X/B} \otimes T_B}$$

Iterated Kodaira-Spencer maps (continued).

$$\rho_f^{(1)} : T_B^{\otimes 2} \to R^1 f_* T_{X/B} \otimes T_B$$

$$\rho_f^{(2)} : R^1 f_* T_{X/B} \otimes T_B \to R^2 f_* T_{X/B}^2$$

$$\boxed{\rho_f^{(2)} \circ \rho_f^{(1)} : T_B^{\otimes 2} \to R^2 f_* T_{X/B}^2}$$

$$\boxed{\rho_g \otimes \rho_h : T_B^{\otimes 2} \to R^1 g_* T_{Y/B} \otimes R^1 h_* T_{Z/B} \simeq R^2 f_* T_{X/B}^2}$$

Proposition. $\rho_f^{(2)} \circ \rho_f^{(1)} = \rho_g \otimes \rho_h$.

Corollary. $\rho_f^{(2)} \circ \rho_f^{(1)} \neq 0 \;\Rightarrow\;$ (**R**) holds for f.

Remark. This statement no longer makes reference to the product structure.

General case: X is no longer a product. $1 \leq p \leq n$,

$$0 \to T^p_{X/B} \otimes f^* T_B^{\otimes(n-p)} \to T^p_X \otimes f^* T_B^{\otimes(n-p)} \to$$

$$\to T^{p-1}_{X/B} \otimes f^* T_B^{\otimes(n-p+1)} \to 0$$

$$\boxed{\rho_f^{(p)} : R^{p-1} f_* T^{p-1}_{X/B} \otimes T_B^{\otimes(n-p+1)} \to R^p f_* T^p_{X/B} \otimes T_B^{\otimes(n-p)}}$$

Definition. $\rho_f := \rho_f^{(n)} \circ \rho_f^{(n-1)} \circ \cdots \circ \rho_f^{(1)}$

$$\boxed{\rho_f : T_B^{\otimes n} \longrightarrow R^n f_* T^n_{X/B}}$$

Definition. f is called *strongly non-isotrivial* if $\rho_f \neq 0$.

Example. Let $Y_i \to B$ be non-isotrivial families of smooth projective curves for $i = 1, \ldots, r$. Then $X = Y_1 \times_B \cdots \times_B Y_r \to B$ is strongly non-isotrivial.

Remark. Since T_B is a line bundle and $R^n f_* T^n_{X/B}$ is locally free, $\rho_f \neq 0$ if and only if it is injective.

The case of $\dim B > 1$**.**

Let (again) $f : X \to B$ be a smooth projective family of varieties (of general type) of dimension n, B a smooth (not necessarily projective) variety.

For an integer p, $1 \leq p \leq n$,

$$T_X^p = \mathscr{F}^0 \supseteq \mathscr{F}^1 \supseteq \cdots \supseteq \mathscr{F}^p \supseteq \mathscr{F}^{p+1} = 0$$

$$\mathscr{F}^i \Big/ \mathscr{F}^{i+1} \simeq T^i_{X/B} \otimes f^* T_B^{p-i}$$

In particular, $\mathscr{F}^p \simeq T^p_{X/B}$ and $\mathscr{F}^{p-1}/\mathscr{F}^p \simeq T^{p-1}_{X/B} \otimes f^* T_B$.

$\rightsquigarrow$

$$0 \to T^p_{X/B} \otimes f^* T_B^{\otimes(n-p)} \to \mathscr{F}^{p-1} \otimes f^* T_B^{\otimes(n-p)} \to$$

$$\to T^{p-1}_{X/B} \otimes f^* T_B^{\otimes(n-p+1)} \to 0$$

$$\boxed{\rho_f^{(p)} : R^{p-1} f_* T^{p-1}_{X/B} \otimes T_B^{\otimes(n-p+1)} \to R^p f_* T^p_{X/B} \otimes T_B^{\otimes(n-p)}}$$

Definition. $\rho_f := \rho_f^{(n)} \circ \rho_f^{(n-1)} \circ \cdots \circ \rho_f^{(1)}$

$$\boxed{\rho_f : T_B^{\otimes n} \longrightarrow R^n f_* T_{X/B}^n}$$

Definition. f is called *strongly non-isotrivial (everywhere) over* B if ρ_f is injective.

Example. Let $Y_i \to B$ be non-isotrivial families of smooth projective curves for $i = 1, \dots, r$. Then $X = Y_1 \times_B \cdots \times_B Y_r \to B$ is strongly non-isotrivial over B.

Remark. One can consider various refinements:

- Considering maps for which the composition of fewer $\rho^{(p)}$'s is injective or non-zero. This is important in particular to study moduli spaces of varieties that products with one rigid term.

- Combining this condition with $\mathrm{Var}(f)$, the variation of f in (birational) moduli.

THEOREM. Let f be a smooth projective family of varieties of general type. If f is strongly non-isotrivial over B, then (**R**) holds for f.

To do. Find more examples of strongly non-isotrivial families.

“Simple Question” revisited. Let $g : Y \to B$ and $h : X \to Y$ be non-isotrivial families of curves, $\dim B = 1$.

$$X \xrightarrow{h} Y \xrightarrow{g} B, \qquad f : X \to B$$

Does (**R**) hold for f?

Remarks.

- The assumption that h is non-isotrivial is not the most natural condition in this situation.

- Over bases of dimension > 1 one usually requires that the variation of the family is maximal. In this case that means $\mathrm{Var}(h) = \dim Y = 2$.

- However, if X is the product of two non-isotrivial families of curves over B, then $\mathrm{Var}(h) = 1$.

$$X \xrightarrow{h} Y \xrightarrow{g} B, \qquad f = g \circ h : X \to B$$

The Kodaira-Spencer map, $\rho_h : T_Y \to R^1 h_* T_{X/Y}$, measures the variation of the family over Y, but we are only interested in variation over B.

Definition. h is *non-isotrivial with respect to* B if

$$\ker \rho_h \subset T_{Y/B}.$$

Lemma. h is *non-isotrivial with respect to* B if and only if $g^* T_B \to R^1 h_* T_{X/B}$ is injective.

Proposition. If g and h are non-isotrivial with respect to B, then f is strongly non-isotrivial over B.

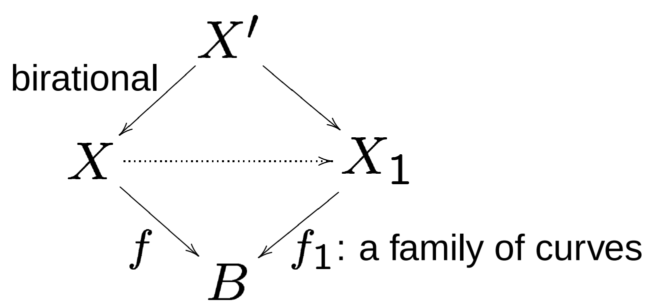
Corollary.
- (**R**) holds for f.
- The answer to the “Simple Question” is indeed yes.

Let $f : X \to B$ be a smooth projective family,

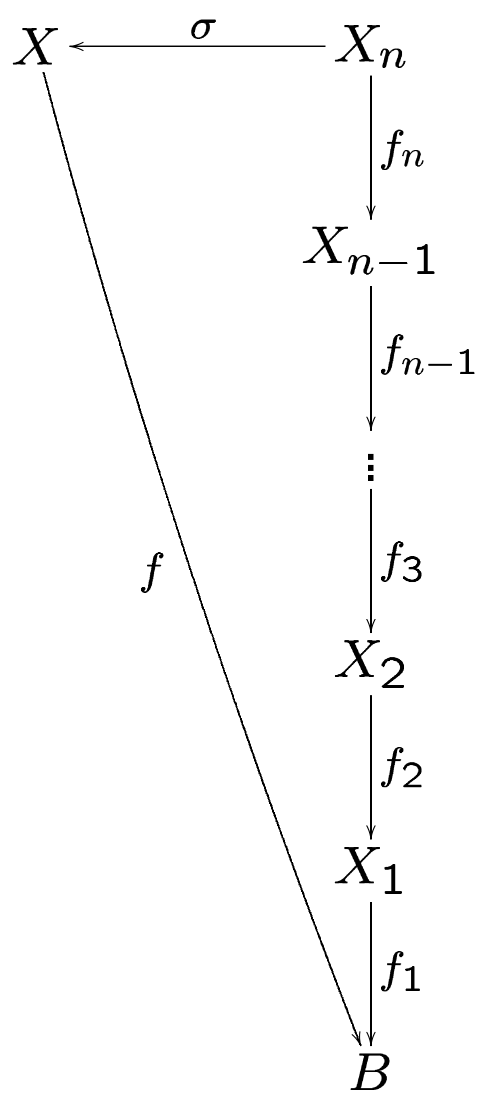
$n = \dim X - \dim B$.

(Weak de Jong) Procedure.

Step One. Take a general projection onto $\mathbb{P}^1_B$ and use Stein factorization. This produces



Step Two. Iterating Step One produces $X_0 = B, X_1, \ldots, X_n$ such that there exists a birational morphism $\sigma : X_n \to X$ and for every $i = 1, \ldots, n$, $f_i : X_i \to X_{i-1}$ is a family of curves.



CONJECTURE

f is strongly non-isotrivial

$$\Updownarrow$$

f_i are non-isotrivial with respect to B for $i = 1, \ldots, n$.

Corollary of conjecture f_i is non-isotrivial with respect to B for $i = 1, \ldots, n$

$$\Downarrow$$

(**R**) holds for f

Lemma. f is strongly non-isotrivial if and only if $f \circ \sigma$ is strongly non-isotrivial.

THEOREM. f_i is smooth and non-isotrivial with respect to B for $i = 1, \ldots, n$

$$\Downarrow$$

f is strongly non-isotrivial