??. AMPLE LINE BUNDLES ON MODULI SPACES

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This chapter is devoted to prove the projectivity of the moduli space of surfaces of general type. Indeed we prove a more general result as follows. (For the definition of semi-positivity see §1.) Most of the ideas appearing in this chapter are already present in [Kollár90] where the matter is discussed in more generality. Here we include a somewhat simplified proof restricting to the necessary cases.

??.0.1 Theorem. *Let $f : X \longrightarrow Y$ be a flat morphism of proper schemes and L a $\mathbb{Q}$-line bundle on X, such that L is f-ample and $f_*(L^k)$ is a semi-positive vector bundle for all k large and sufficiently divisible. Suppose every fibre is isomorphic to only finitely many others, and for a fibre X_y there are only finitely many automorphisms of X_y fixing $L^k|_{X_y}$.*

Then $\det f_*(L^m)$ *is ample, for m large and sufficiently divisible. In particular Y is projective.*

??.0.0.1 Remark. The same statement is true for algebraic spaces with essentially the same proof.

§??.1. Semi-positive vector bundles

??.1.1 Definition. Let V be a vector bundle over a scheme Y. We say that V is *semi-positive*, if for every morphism $g : C \longrightarrow Y$ from a smooth projective curve C and every quotient bundle Q of g^*V, $c_1(Q) \geq 0$.

??.1.2 Lemma. *Let V be a vector bundle over a scheme Y. The following are equivalent.*

(1) *V is semi-positive.*
(2) *For every morphism $g : C \longrightarrow Y$ from a smooth projective curve C and every quotient line bundle L of g^*V, $c_1(L) \geq 0$.*
(3) *$\mathcal{O}_{\mathbb{P}(V)}(1)$ is nef on $\mathbb{P}(V)$.*
(4) *For every morphism $g : C \longrightarrow Y$, $g^*V \otimes H$ is ample, where H is ample on C.*

Proof. Clearly (1) implies (2). Let $\pi : \mathbb{P}(V) \longrightarrow Y$ be the natural projection. Let $C \subset \mathbb{P}(V)$ be a curve. Then $\mathcal{O}_{\mathbb{P}(V)}(1)|_C$ is a quotient of $\pi^* V|_C$. Thus (2) implies (3).

Let $g : C \longrightarrow Y$ be any morphism. Then we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{P}(g^*V) & \xrightarrow{\tilde{g}} & \mathbb{P}(V) \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ C & \xrightarrow[g]{} & Y \end{array}$$

such that $\mathcal{O}_{\mathbb{P}(g^*V)}(1) = g^*\mathcal{O}_{\mathbb{P}(V)}(1)$. Now if $\mathcal{O}_{\mathbb{P}(V)}(1)$ is nef, then so is $\mathcal{O}_{\mathbb{P}(g^*V)}(1)$ and then for any ample line bundle H on C, $\mathcal{O}_{\mathbb{P}(g^*V)}(1) \otimes \tilde{\pi}^* H$ is ample. This is exactly the ampleness of $g^*V \otimes H$ (cf. [Hartshorne66, 3.2]). Hence (3) implies (4).

A quotient bundle of an ample vector bundle is ample and has positive degree [Hartshorne66, 2.2 and 2.6]. Now choose H ample of degree one and conclude that any line bundle quotient L of g^*V has degree at least $1 - \deg H = 0$. Thus (4) implies (2).

Let $g : C \longrightarrow Y$ be a morphism, and Q a quotient bundle of g^*V. Let $W = \mathbb{P}(Q)$ be the subscheme of $\mathbb{P}(g^*V)$ corresponding to Q. Then $c_1(Q) = c_1(g^*\mathcal{O}_{\mathbb{P}(V)}(1))^k|_W$, where k is the dimension of W. Thus (3) implies (1). $\square$

??.1.2 Lemma. *Let V be a semi-positive vector bundle. Then $W = \operatorname{Sym}^d(V)$ is semi-positive.*

Proof. There are several ways to prove this result. We present a very general one given in [Kollár90].

First assume the characteristic of the groundfield is not zero. By definition of semi-positivity, we may assume Y is a curve. Note that $H^1(Y, V \otimes L)$ vanishes, for $c_1(L) \geq 2g - 1$, since every map from $V \otimes L$ to ω_Y must be zero. Thus $V \otimes L$ is globally generated, for $c_1(L) \geq 2g + 1$ and we have a surjective map:

$$\bigoplus_{i=1}^{r} L^{-1} \to V.$$

It follows that W is a quotient of the direct sum of line bundles whose degree is bounded from below by the constant $N = -d(2g + 1)$. Note that N depends only on d and on the genus of the curve, but not on V.

Suppose Q is a quotient vector bundle of W, of negative degree. Applying base change by powers of the Frobenius, we obtain V', W' and Q', such that $c_1(Q') < N$, a contradiction.

To finish the proof in characteristic zero we use the reduction mod p technique (cf. [Kollár94, II.5.10]).

Suppose W is not semi-positive, i.e., it has a quotient bundle of negative degree. This remains true after reducing mod p for an open dense set of primes. After base change we may assume, that W reduced mod p has a quotient bundle of degree less than $-d$ for an open dense set of primes.

Now choose an ample line bundle of degree 1. $V \otimes H$ is ample, ampleness is an open condition in flat families, so $V \otimes H$ is ample mod p for an open dense set of primes. Then by the above argument $\mathrm{Sym}^d(V \otimes H) \simeq W \otimes H^{\otimes d}$ is semi-positive, so every quotient bundle of W has degree at least $-d$, a contradiction. □

§??.2. Ampleness Theorem

We now turn to the proof of (??.0.0). We review the general idea of the proof. First choose the integers k and d satisfying the following properties:

1. L^k is f-very ample
2. $R^i f_*(L^{j\cdot k}) = 0$ for $i, j > 0$
3. the multiplication map $p : \mathrm{Sym}^d(f_*(L^k)) \to f_*(L^{d\cdot k})$ is surjective
4. every fibre X_y, embedded into the projective space $\mathbb{P}_y$ via $L^k|_{X_y}$ is defined (set theoretically) by degree d equations.

Set $V = f_*(L^d)$, and $W = \mathrm{Sym}^d(V)$. Then $Q = f_*(L^m)$ is a quotient of W for $m = dk$. Now Q is semi-positive and so is $\det Q$ by (??.1.1). We claim that $\det Q$ is ample. It remains to use the fact that f moves in moduli as much as possible.

Suppose V has rank r and let $G = \mathrm{GL}(r)$. Clearly W has structure group G.

??.2.1 Definition. Let W be a vector bundle of rank n with structure group G and $p : W \longrightarrow Q$ a quotient bundle of rank l. Let $\mathrm{Gr}(l, n)/G$. denote the set of G-orbits on the l-dimensional quotients of an n-dimensional vector space. The *classifying map* is the natural map of sets $u : Y \longrightarrow \mathrm{Gr}(l, n)/G$.

We will say that u is *finite*, if

1. the fibres of u are all finite, and
2. the fibres of p have finite G-stabilisers.

Note that in our case the classifying map is automatically finite. Indeed, if y is a point of Y, the embedding of X_y into $\mathbb{P}_y$ identifies $H^0(X_y, L^k|_{X_y}) \simeq H^0(\mathbb{P}_y, \mathcal{O}_{\mathbb{P}_y}(1))$ in a natural way. Then p at the point y is the same as the restriction map, i.e., we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Sym}^d(H^0(X_y, L^k|_{X_y})) & \longrightarrow & H^0(X_y, L^{d\cdot k}|_{X_y}) \\ \downarrow \simeq & & \downarrow \simeq \\ H^0(\mathbb{P}_y, \mathcal{O}_{\mathbb{P}_y}(d)) & \longrightarrow & H^0(X_y, \mathcal{O}_{\mathbb{P}_y}(d)|_{X_y}) \end{array}$$

Therefore the kernel of p consists of the degree d polynomials cutting out X_y under the embedding given by L^k. Let $y_1, y_2 \in Y$ be such that $u(y_1) = u(y_2)$. Then there is an element of G that maps the kernel of p at y_1 isomorphically to the kernel of p at y_2. Hence it induces an isomorphism between X_{y_1} and X_{y_2}. Since every fibre is isomorphic to only finitely many others this implies that u has finite fibres.

Similarly, the fibres of p have finite G-stabilisers since for every X_y there are only finitely many automorphisms of X_y fixing $L^k|_{X_y}$.

Thus we are reduced to proving the following result.

??.2.2 Theorem. *Let V be a semi-positive vector bundle over the scheme Y, and suppose Q is a quotient of $W = \operatorname{Sym}^d V$, such that the classifying map is finite.*

Then $\det Q$ *is ample.*

To prove (??.2.1) we are going to apply the Nakai-Moishezon criteria (cf. [Nakai 63], [Moishezon67] and [Kleiman66]):

??.2.2 Theorem (Nakai-Moishezon). *Let H be a line bundle on the proper scheme Y. Suppose that $(c_1(H|_Z))^{\dim Z} > 0$, for every closed irreducible subvariety Z of Y.*

Then H is ample.

Pick a subvariety Z of Y. By Chow's Lemma, if we normalise Z, and blow it up, then we may make Z projective, but we lose the fact that the classifying map is finite. Thus we are reduced to proving:

??.2.2 Lemma. *Let V be a semi-positive vector bundle over the projective variety Y, and suppose Q is a quotient of $W = \operatorname{Sym}^d V$, such that the classifying map is finite over an open subset of Y.*

Then $c_1(Q)^{\dim Y} > 0$.

Proof. We introduce a projective bundle $\mathbb{P}$ over Y, so that we may lift the classifying map to a rational map $g : \mathbb{P} \longrightarrow \operatorname{Gr}(l, n)$. Set

$$\mathbb{P} = \mathbb{P}(\bigoplus_{i=1}^{r} V^*)$$

and let π be the natural projection to $\mathbb{P} \longrightarrow Y$. Consider the natural map of (3(!!).7.7)

$$t : \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \pi^* V.$$

It is generically surjective and by composition if induces a generically surjective map

$$\theta : \operatorname{Sym}^d(\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}}(-1)) \longrightarrow \pi^* Q$$

Replacing $\mathbb{P}$ by a blow up $b : \mathbb{P}' \longrightarrow \mathbb{P}$, we may assume that $b^*\theta$ surjects onto a locally free subsheaf $\mathcal{B}$ of $b^*\pi^*Q$.

Now we consider the following twist of $b^*\theta$

$$\theta' : \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}'} \simeq b^*(\operatorname{Sym}^d(\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}}(-1)) \otimes \mathcal{O}_{\mathbb{P}}(d)) \longrightarrow b^*(\pi^* Q \otimes \mathcal{O}_{\mathbb{P}}(d)).$$

The classifying map of this quotient is a morphism

$$g : \mathbb{P}' \longrightarrow \mathrm{Gr}(l, n).$$

which is a lifting of the classifying map $u : Y \longrightarrow \mathrm{Gr}(l,n)/G$.

For simplicity we assume that G is transitive on the fibres of $\mathbb{P}$ over Y. This is the case we need for our application. For the general case see [Kollár90, 3.13]. Now g is generically finite, since the classifying map u is finite over an open subset of Y, and $\mathbb{P}$ has structure group G. Let $\mathcal{O}_{\mathrm{Gr}(l,n)}(1)$ be the very ample line bundle on $\mathrm{Gr}(l,n)$ giving the Plücker embedding. Then $\det(\mathcal{B}) \otimes b^*\mathcal{O}_{\mathbb{P}}(dl) = g^*\mathcal{O}_{\mathrm{Gr}(l,n)}(1)$ is big.

Pick an ample divisor H on Y. For e large enough, $g^*\mathcal{O}_{\mathrm{Gr}(l,n)}(e) \otimes b^*\pi^*\mathcal{O}_{\mathbb{P}}(-H)$ will have a section. Thus there is a non-trivial map

$$\mathcal{O}_{\mathbb{P}'} \longrightarrow g^*\mathcal{O}_{\mathrm{Gr}(l,n)}(e) \otimes b^*\pi^*\mathcal{O}_Y(-H) \simeq \det(\mathcal{B})^{\otimes e} \otimes b^*(\mathcal{O}_{\mathbb{P}}(edl) \otimes \pi^*\mathcal{O}_Y(-H))$$

Composing with the natural map $\det(\mathcal{B}) \longrightarrow \det(Q)$, pushing forward to $\mathbb{P}$ and then to Y yields

$$\begin{aligned} \mathcal{O}_{\mathbb{P}'} &\longrightarrow b^*(\det(Q)^{\otimes e} \otimes \mathcal{O}_{\mathbb{P}}(edl) \otimes \pi^*\mathcal{O}_Y(-H)) \\ \mathcal{O}_{\mathbb{P}} &\longrightarrow \pi^*\det(Q)^{\otimes e} \otimes \mathcal{O}_{\mathbb{P}}(edl) \otimes \pi^*\mathcal{O}_Y(-H) \\ \mathcal{O}_Y &\longrightarrow \det(Q)^{\otimes e} \otimes \pi^*\mathcal{O}_Y(-H) \otimes \pi_*(\mathcal{O}_{\mathbb{P}}(edl)). \end{aligned}$$

Finally we have a nontrivial map

$$\phi : (\pi_*(\mathcal{O}_{\mathbb{P}}(edl)))^* \simeq \mathrm{Sym}^{edl}(\bigoplus_{i=1}^{r} V) \longrightarrow \det(Q)^{\otimes e} \otimes \pi^*\mathcal{O}_{Y(-H)}$$

Replacing Y by a blow up, we may assume the image of ϕ is a locally free subsheaf $\mathcal{G}$. $c_1(\mathcal{G})$ is nef since $\mathrm{Sym}^{edl}(\bigoplus_{i=1}^{r} V)$ is semi-positive. Thus we have

$$e \cdot c_1(Q) = E + c_1(\mathcal{G}) + H,$$

where E is an effective divisor, $c_1(\mathcal{G})$ is nef, H is ample and $c_1(Q)$ is nef. Then

$$e^{\dim Y} \cdot c_1(Q)^{\dim Y} = H^{\dim Y} + \sum_{i=1}^{\dim Y} H^{(\dim Y - i)} \cdot c_1(Q)^{(i-1)} \cdot (E + c_1(\mathcal{G})) > 0. \quad \square$$

This completes the proof of (??.2.1) and (??.0.0).

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