Algebra = Geometry

Sándor Kovács
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Motto

"To me, algebraic geometry is algebra with a kick"

–Solomon Lefschetz

Geometry

- Geometry = Space + Functions

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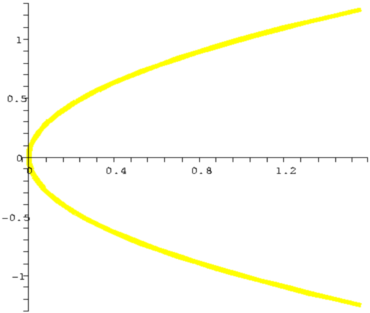

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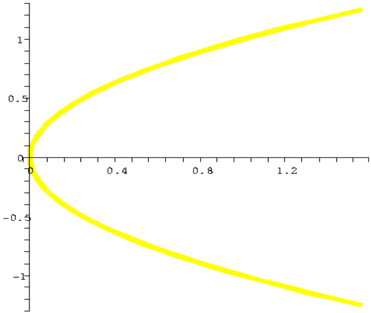

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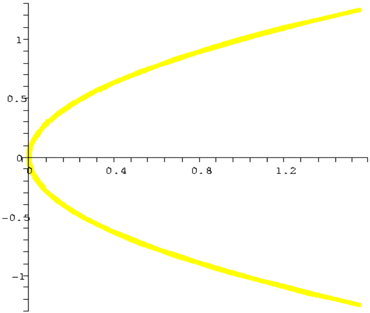

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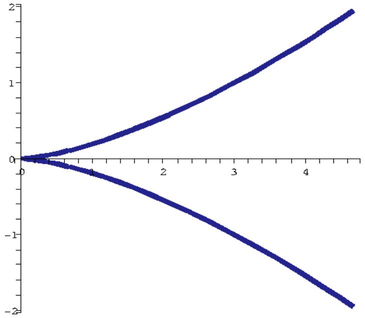

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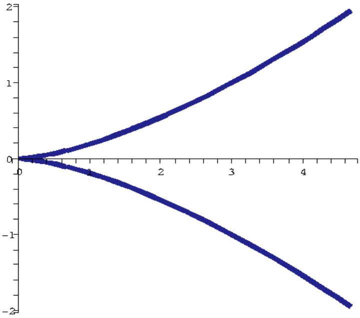

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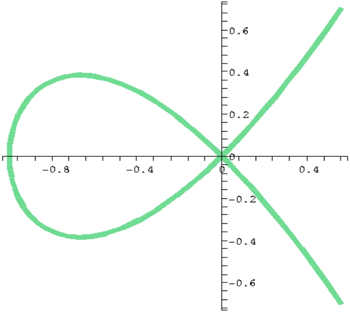

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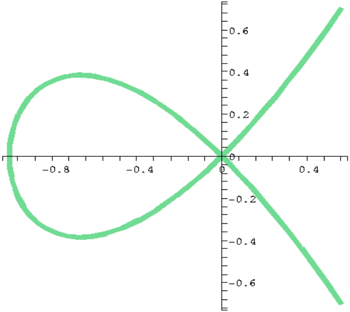

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$$\boxed{\{\text{affine varieties}\} \leftrightarrow \{\text{finitely generated } \mathbb{C}\text{-algebras}\}.}$$

Curves

Complex Projective Curve

$=$ Riemann Surface

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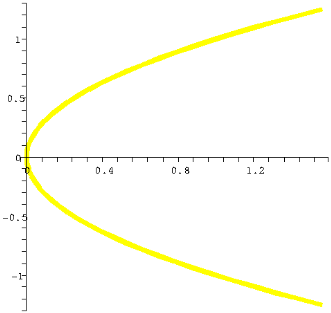

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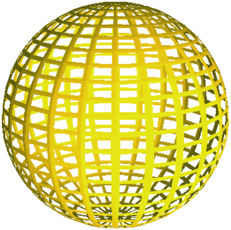

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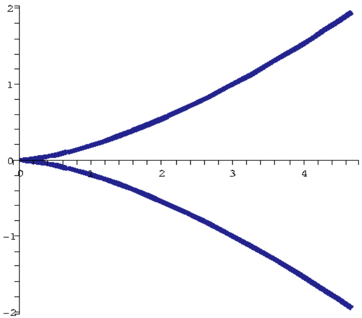

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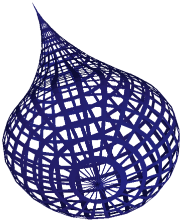

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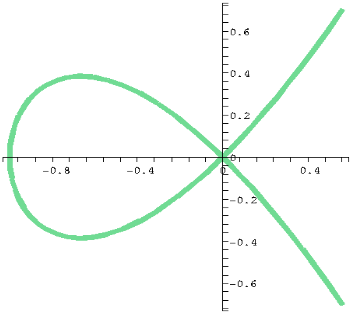

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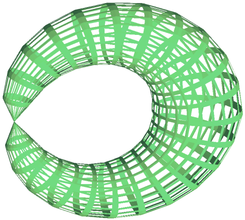

Rational Curves

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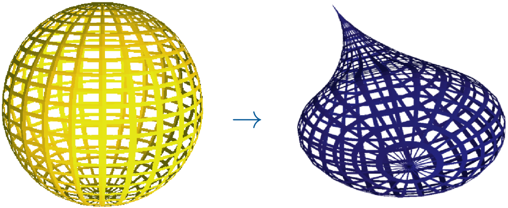

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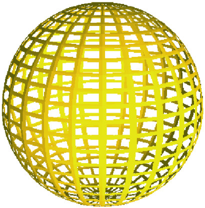
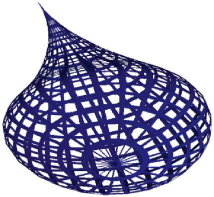
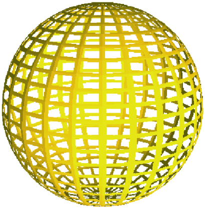
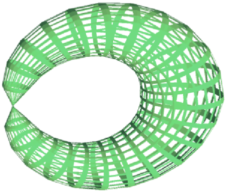
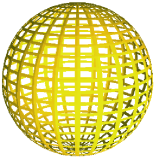
 → 

 → 

Genus



genus 0

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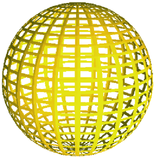

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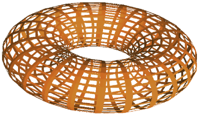

genus 1

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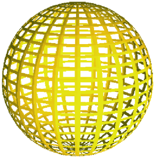
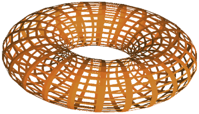


genus 0



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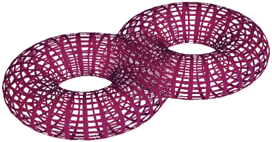
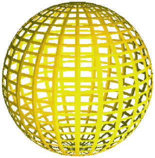


genus 2

Genus



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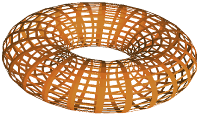

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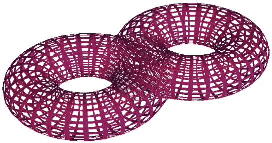

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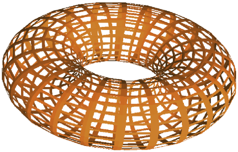

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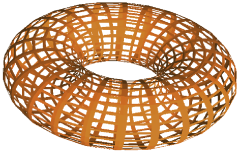

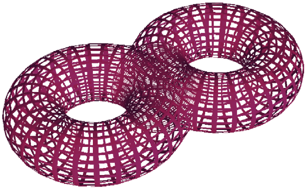

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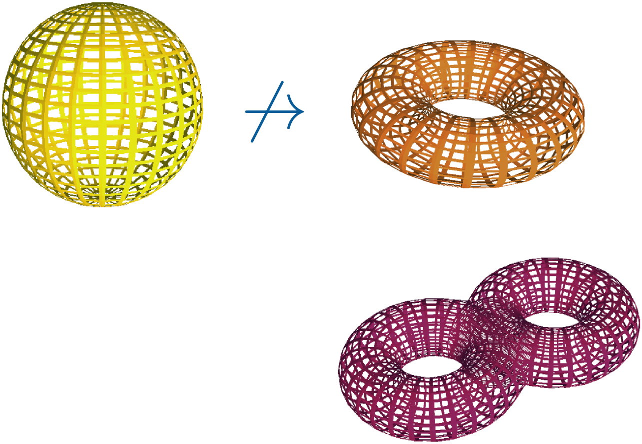

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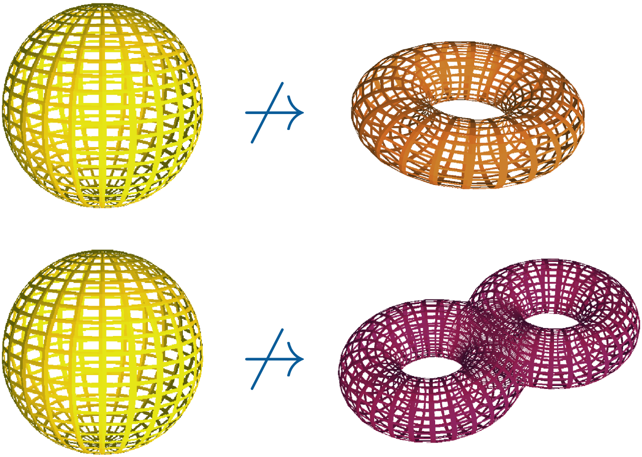

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- In particular, any two curves are homeomorphic.

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- $\mathscr{O}_{X,P} \subseteq K(X)$ subring.

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Example

- Let $X = \mathbb{C}$. Then $A(X) = \mathbb{C}[t]$, $K(X) = \mathbb{C}(t)$.
- Let $P = 0 \in X$. Then

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- In the previous example, v_P is a valuation of $K(X)$, and $R_{v_P} = \mathscr{O}_{X,P}$.

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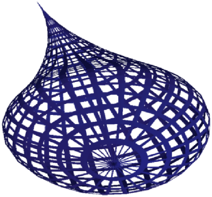
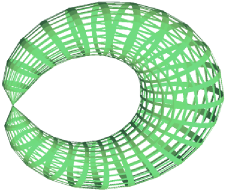
HOMEWORK:
Let $X = (y^2 = x^3) \subset \mathbb{C}^2$, $P = (0,0) \in X$.
Prove that $\mathcal{O}_{X,P}$ is **not** a valuation ring of $K(X)$.

Singularities





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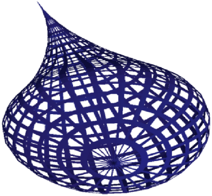

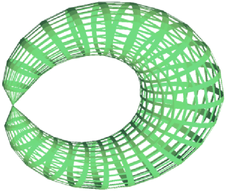

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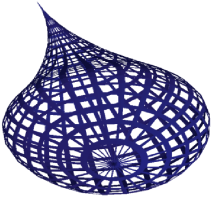

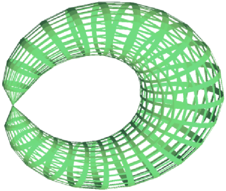

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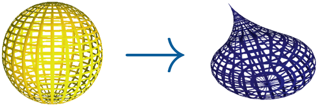

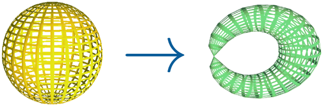

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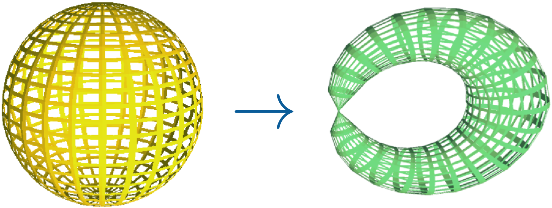

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- Find $\tilde{X}$ smooth, such that $K(\tilde{X}) \simeq K(X)$.

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- More generally, if $R_v \subset K$ is a valuation ring, then $\mathfrak{m}_v = \{f \in R_v | v(f) > 0\}$ is a maximal ideal and $R_v/\mathfrak{m}_v \simeq \mathbb{C}$.

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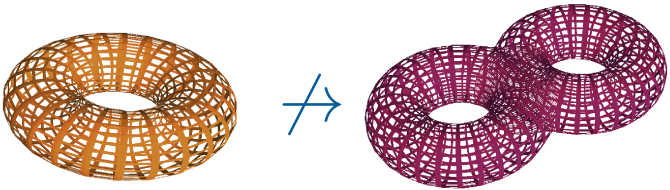

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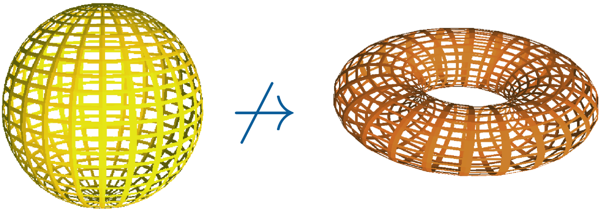

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- Therefore $L \simeq K(X_L) \simeq K(\mathbb{P}^1) \simeq \mathbb{C}(t)$.

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- The similar statement holds in dimension two:
  A unirational surface is rational, or analogously if
  $\mathbb{C} \subsetneq L \subseteq \mathbb{C}(t, u)$, then L is purely transcendental.

Lüroth Problem

- Geometric Version: A unirational curve is rational.
- What about higher dimensional varieties?
- The similar statement holds in dimension two:
  A unirational surface is rational, or analogously if $\mathbb{C} \subsetneq L \subseteq \mathbb{C}(t, u)$, then L is purely transcendental.
- Iskovskih-Manin (1971), Clemens-Griffiths (1972), Artin-Mumford (1972):
  There exist three-dimensional unirational but not rational varieties, such as a some cubic and quartic hypersurfaces in $\mathbb{P}^4$.

Higher Dimensions

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- Morin (1940), Predonzan (1949): There exist a computable function ν such that if $n \geq \nu(d)$, then X is unirational.
- Kollár (1995): If $d \geq \frac{2}{3}(n+4)$, and X is general, then it is not rational.

Open Problems

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- Let $n = 4$ and $d = 3$.
  Is the general cubic fourfold non-rational?
  It is known that they are unirational.

The End

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`http://latex-beamer.sourceforge.net`