# Families of varieties of general type: the Shafarevich conjecture and related problems

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# Motto

*"To me,*
*algebraic geometry*
*is algebra with a kick"*

–Solomon Lefschetz

# Outline

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1. Curves
   - Riemann Surfaces
   - Genus
   - Topology, Arithmetic and Differential Geometry

Smooth Complex Projective Curve

=

Compact Riemann Surface

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   - Topology, Arithmetic and Differential Geometry

# Genus

genus 0

genus 1

genus 2, ...

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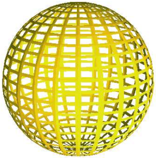

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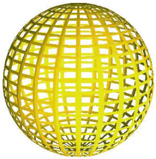

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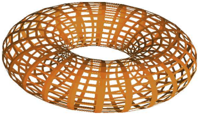

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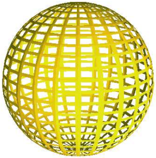

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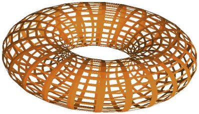

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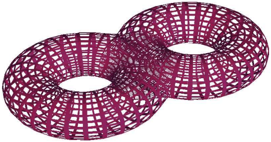

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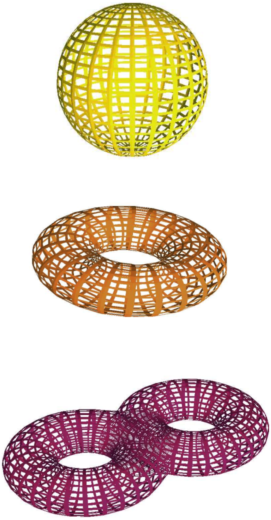

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# Differential Geometry of Curves





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- $C$ is a plane cubic elliptic
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# Outline





2. Families
   - Examples and Definitions
   - Isotrivial and Non-isotrivial Families

# A family of curves of general type

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$$\mathbb{A}^2_{x,y} \supseteq X_\lambda = \pi^{-1}(\lambda) \subseteq X$$

$$\downarrow \pi$$

$$\lambda \in \mathbb{A}^1_\lambda$$

# Notation

- $B$ – smooth projective curve of genus $g$,
  $\Delta \subseteq B$ – finite set of points.

- A family over $B \setminus \Delta$, $f : X \to B \setminus \Delta$, is a surjective (flat) map with equidimensional, connected fibers.
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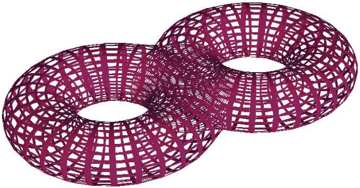

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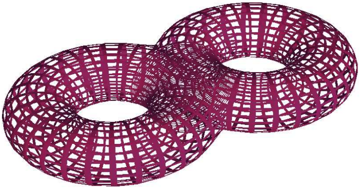

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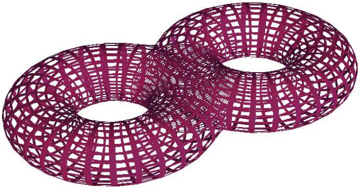

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2 Families

- Examples and Definitions
- Isotrivial and Non-isotrivial Families

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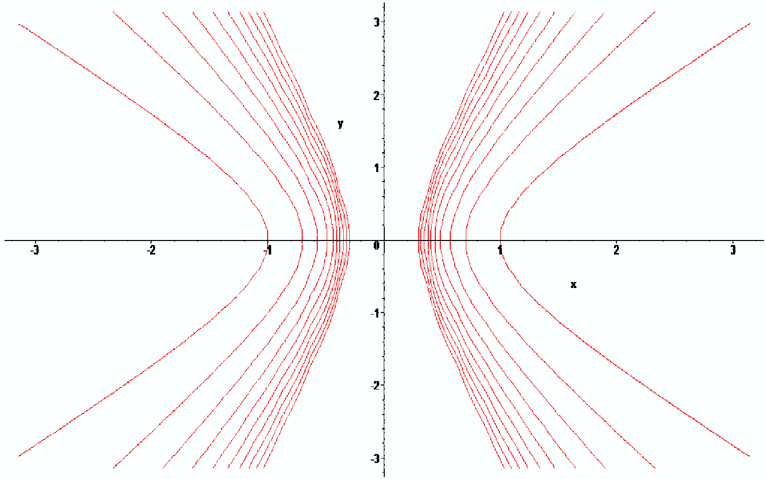

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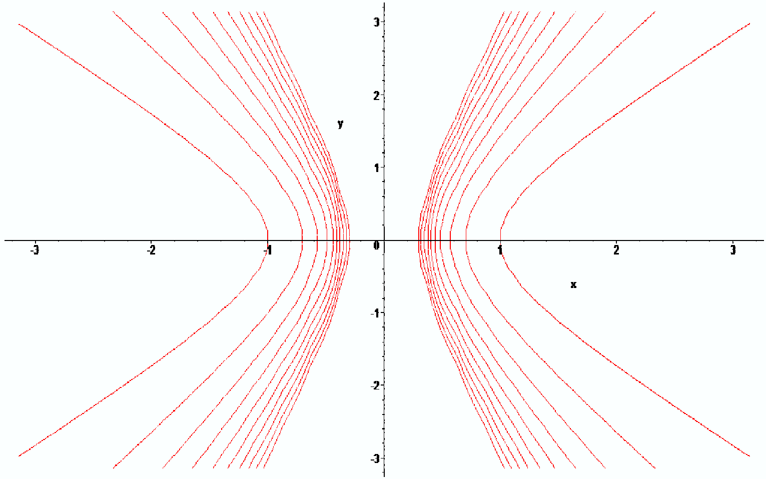


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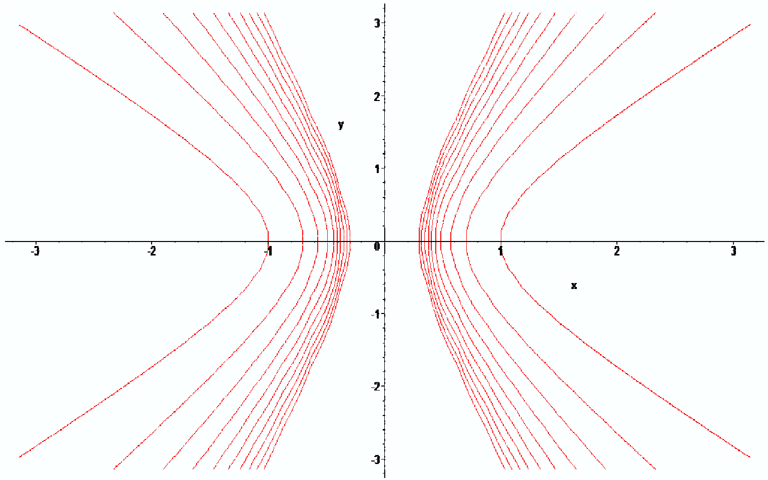


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3 Shafarevich's Conjecture
- Statement and Definitions
- Results and Connections

# Shafarevich's Conjecture (1962)



- Fix $B, \Delta$ as before, $q \in \mathbb{Z}$, $q \geq 2$. Then

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- Note:

$$2g(B) - 2 + \#\Delta \leq 0 \Leftrightarrow \begin{cases} g(B) = 0 & \& \quad \#\Delta \leq 2 \\ g(B) = 1 & \& \quad \Delta = \emptyset \end{cases}$$

# Hyperbolicity

- A complex analytic space $X$ is Brody hyperbolic if every $\mathbb{C} \to X$ holomorphic map is constant.
- $X$ Brody hyperbolic implies that
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- An algebraic variety $X$ is algebraically hyperbolic if
  - $\forall\ \mathbb{A}^1 \setminus \{0\} \to X$ regular map is constant, and
  - $\forall\ A \to X$ regular map is constant, for any abelian variety (projective algebraic group) $A$.

# Hyperbolicity: Examples

- $B$ – smooth projective curve,
  $\Delta \subseteq B$ – finite subset.
  - If $g(B) \geq 2$, then
    $B \setminus \Delta$ is hyperbolic for arbitrary $\Delta$.
  - If $g(B) = 1$, then
    $B \setminus \Delta$ is hyperbolic $\Leftrightarrow \Delta \neq \emptyset$.
  - If $g(B) = 0$, then
    $B \setminus \Delta$ is hyperbolic $\Leftrightarrow \#\Delta \geq 3$.
- $B \setminus \Delta$ is hyperbolic $\Leftrightarrow 2g(B) - 2 + \#\Delta > 0$.

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- Fix $B, \Delta$ as before, $q \in \mathbb{Z}$, $q \geq 2$. Then

(I) there exists only finitely many non-isotrivial smooth families of curves of genus $q$ over $B \setminus \Delta$. "admissible families".

(II) If

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then there aren't any admissible families.

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then there aren't any admissible families.

(II$^*$) If there exists any admissible families, then $B \setminus \Delta$ is hyperbolic.

# Outline

3. Shafarevich's Conjecture
   - Statement and Definitions
   - Results and Connections

# History

- The Shafarevich Conjecture was confirmed by PARSHIN (1968) for $\Delta = \emptyset$, and by ARAKELOV (1971) in general.
- INTERMEZZO: The number field case.
  - *Mordell Conjecture (1922)*:
    $F$ – a number field (a finite extension field of $\mathbb{Q}$),
    $C$ – a smooth projective curve of genus $\geq 2$
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- Let $f : X \to B$ be a non-isotrivial family of projective curves of genus $\geq 2$.
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# Outline

4 The Proof of Shafarevich's Conjecture
- Deformation Theory
- The Arakelov-Parshin method

# Deformations

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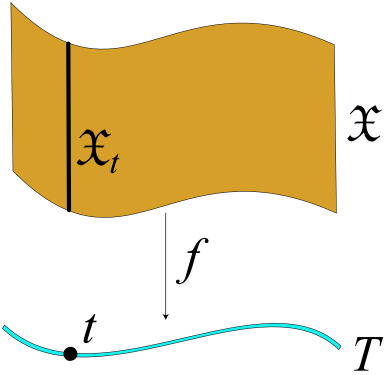

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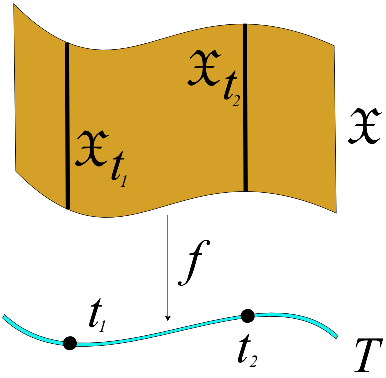

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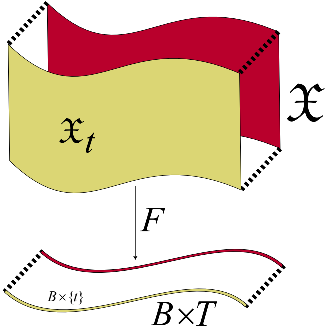

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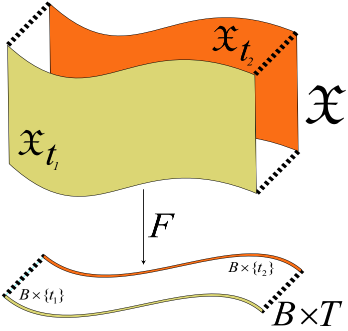

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# Outline

5 Generalizations

- Higher Dimensional Fibers
- Kodaira Dimension
- Admissible Families
- Weak Boundedness
- Recent Results
- Higher Dimensional Bases
- Rigidity

# Higher dimensional fibers

- First order of business:
- Generalize the *setup*.
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  - Hilbert polynomial, $h$.
  - Kodaira dimension, $\kappa = -\infty, 0, 1, \ldots, \dim$.

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Shafarevich Conjecture

Sándor Kovács

Curves

Families

Shafarevich's Conjecture

The Proof

Generalizations

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# Outline

5 Generalizations
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“modern” = last 25 years

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# Outline

5 Generalizations
- Higher Dimensional Fibers
- Kodaira Dimension
- Admissible Families
- Weak Boundedness
- Recent Results
- Higher Dimensional Bases
- Rigidity

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- K___ (1997): If $f : X \to B$ is admissible and $B$ is an abelian variety, then $\Delta \neq \emptyset$.

# Rigidity

But, what about Rigidity?

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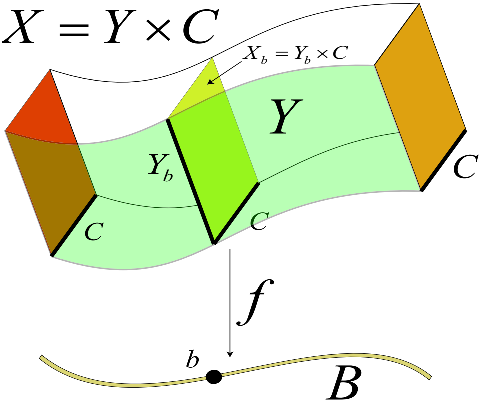

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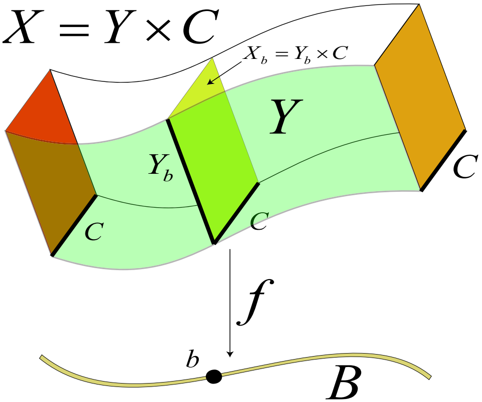

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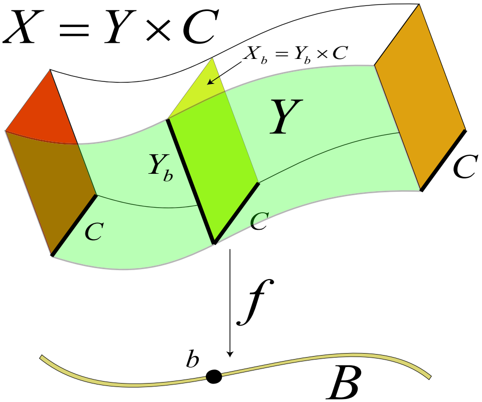

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# Acknowledgement



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`http://latex-beamer.sourceforge.net`