Singularities

Sándor Kovács

May 1, 2007


Field and Galois Theory
Using Algebraic Geometry
BOURBAKI
ELLIPTIC CURVES

First Impressions

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Pierre de Fermat (1601 - 1665)

First Impressions

- Fermat:

$$a^n + b^n = c^n$$

for $n \geq 3$ has no solution with a, b, c non-zero integers.

First Impressions

Bolyai, János (1802 - 1860)

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Évariste Galois (1811 - 1832)

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- Galois: solving equations, group theory, field extensions.

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  - Neither Bolyai nor Galois was recognized by their contemporaries.
  - Galois died at a very young age (21).

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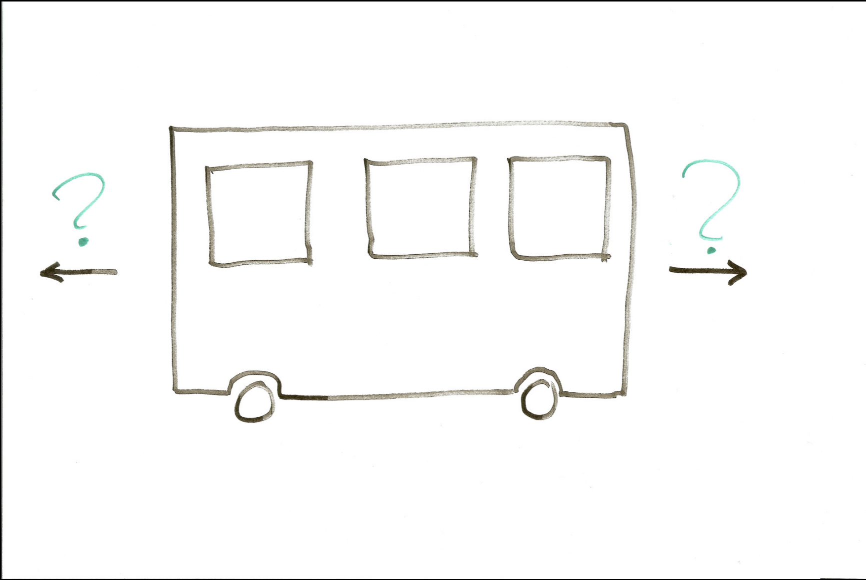
- The Japanese kindergarten entry exam...

The Bus Puzzle



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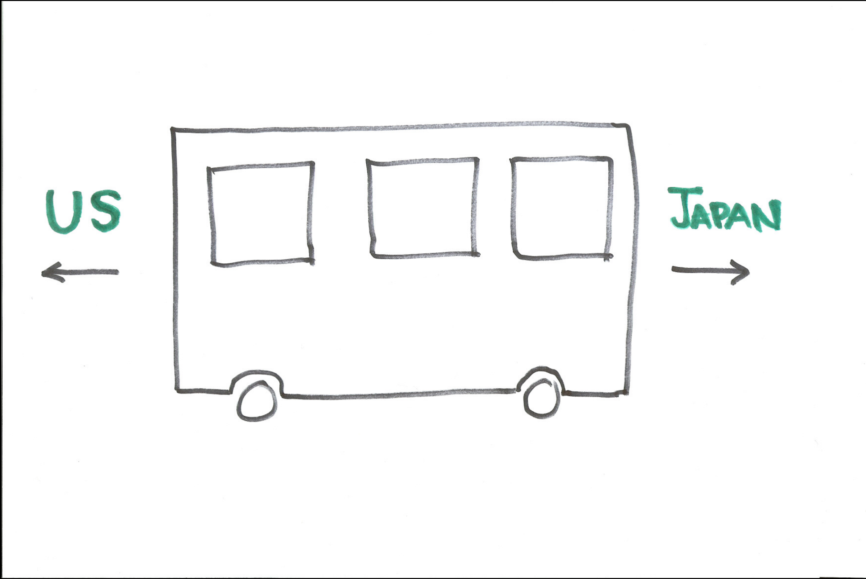

A Few Good Hungarians

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Riesz, Frigyes (1880 - 1956)

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Fejér, Lipót (1880 - 1959)

A Few Good Hungarians

Haar, Alfréd (1885 - 1933)

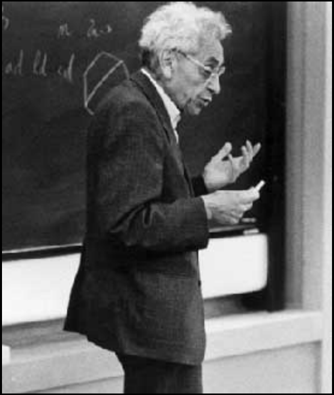
A Few Good Hungarians

Neumann, János (1903 - 1957)

A Few Good Hungarians

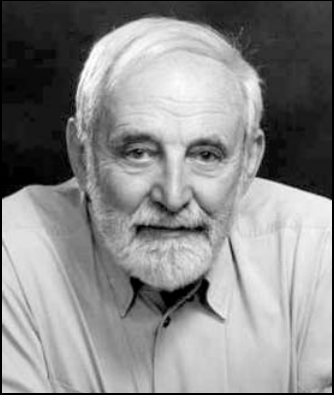
Péter, Rózsa (1905 - 1977)

A Few Good Hungarians



Erdős, Pál (1913 - 1996)

A Few Good Hungarians



Bott, Raoul (1923 - 2005)

...and a few more

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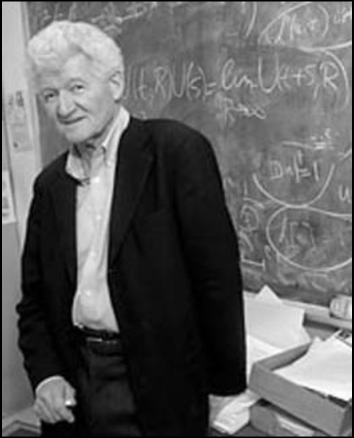
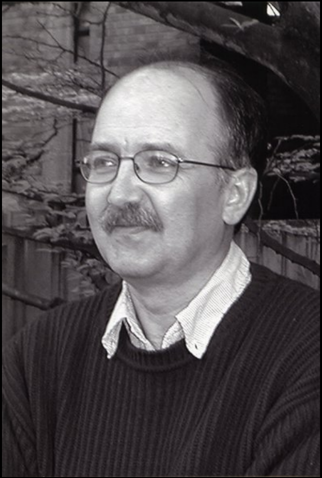


Lax, Péter (1926 -)

...and a few more

Lovász, László (1948 -)

...and a few more



Kollár, János (1956 -)

Mathematical Impressions

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$\rightsquigarrow$ **algebraic geometry**

First Papers

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- Second paper $\longrightarrow$ Erdős$\# = 2$

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- Someone's Erdős# is n if they published a research paper with someone who has $\text{Erdős\#} = n - 1$, but have not published paper with anyone who has $\text{Erdős\#} < n - 1$.

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Example

My Erdős# $= 2$.

Thesis

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"Hey, man, what's your thesis about?"

Thesis

"Well, it has something to do with how the universe is changing through time, and it states that either the universe does not change at all, or there must be times when black holes exist."

Thesis

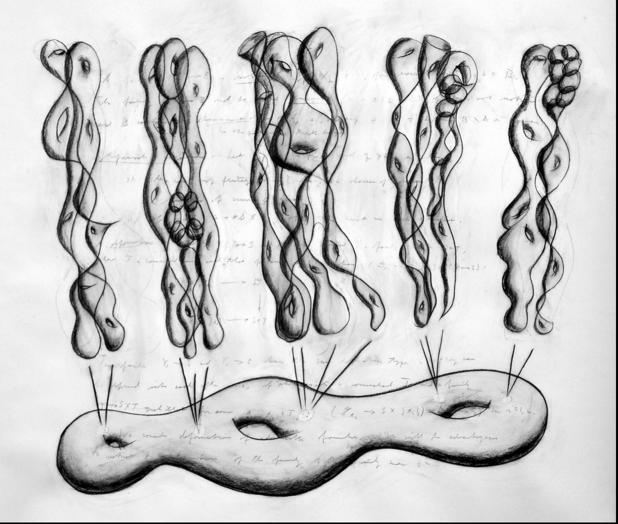
"Well, it has something to do with how the universe is changing through time, and it states that either the universe does not change at all, or there must be times when black holes exist."

(This is, of course, a very loose and non-rigorous interpretation.)

Thesis

"Well, it has something to do with how the universe is changing through time, and it states that either the universe does not change at all, or there must be times when singularities exist."

Thesis



My thesis through Lun Yi's eyes

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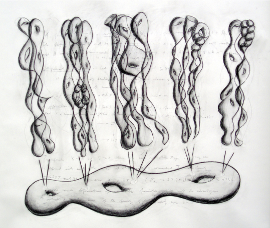

Shift hours: Friday & Saturday 12-5 pm
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www.shiftstudio.org

Above: *Shafarevich's Conjecture*, 2007, charcoal on paper, 35 x 40 in
Front: *Study for Invention*, 2007
Photo credit: Connie Wellnitz

Conics

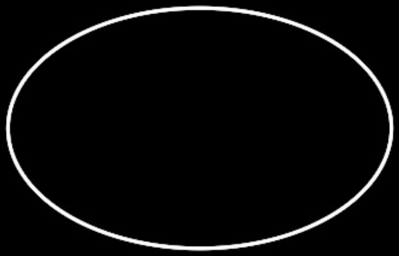
Conics

"This may not be your thesis, but this I understand."

Conics

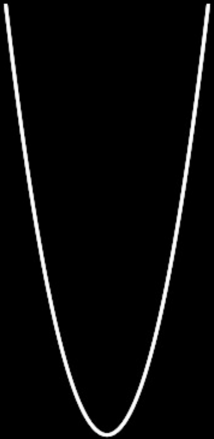
"What are conics, and why are they called "conics"?"

Conics



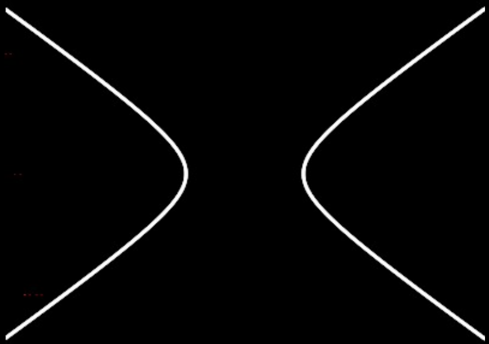
ellipse

Conics



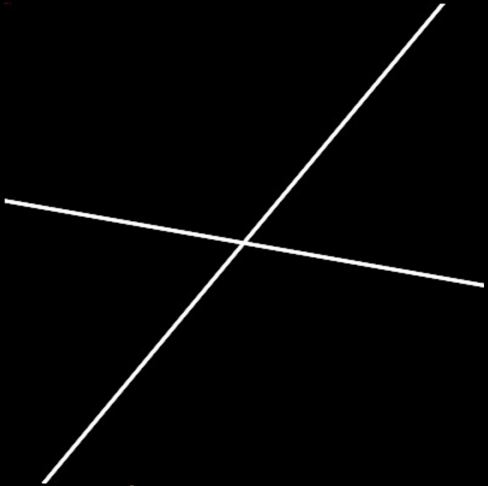
parabola

Conics



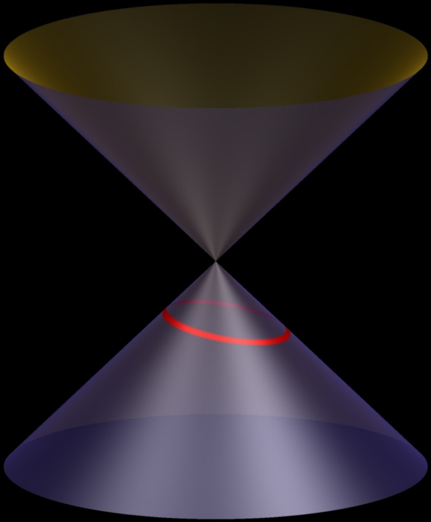
hyperbola

Conics



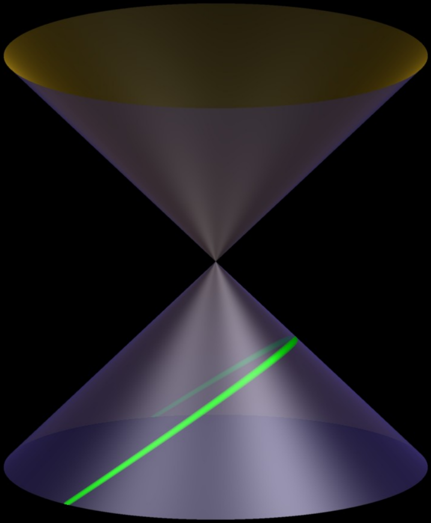
degenerate

Conics



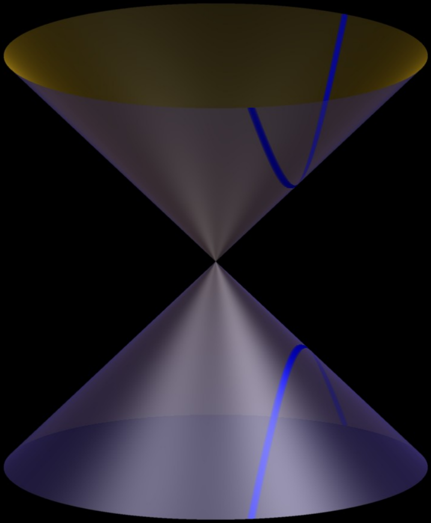
ellipse

Conics



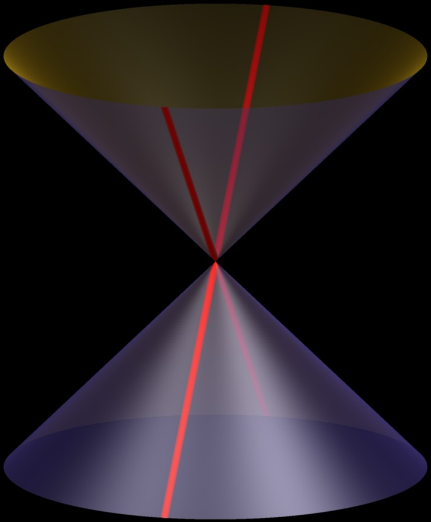
parabola

Conics



hyperbola

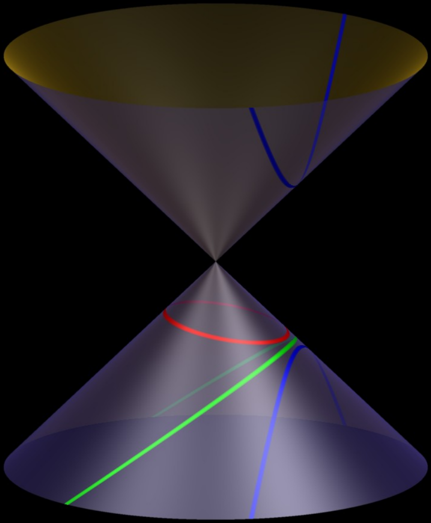
Conics



degenerate

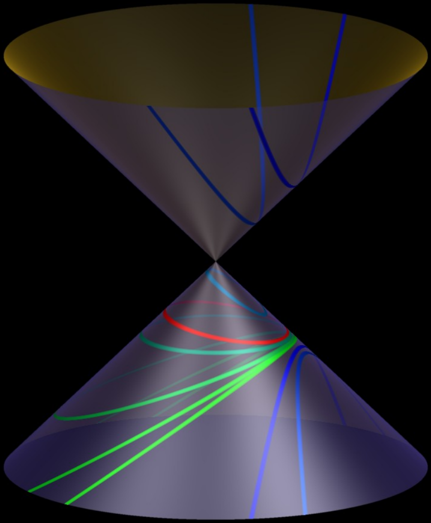
DEFORMATIONS

Conics



deformations

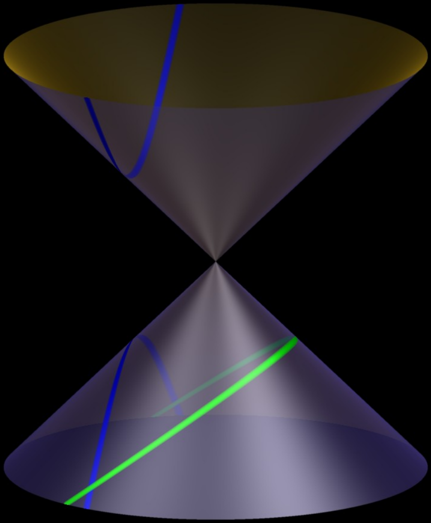
Conics



deformations

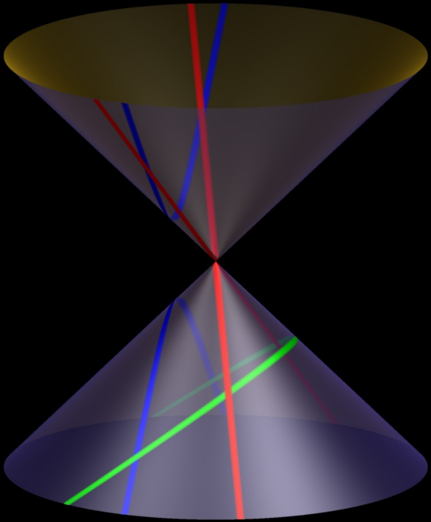
INTERSECTIONS

Conics



intersections

Conics



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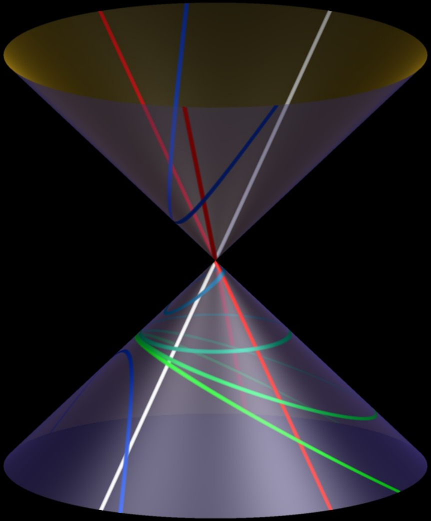
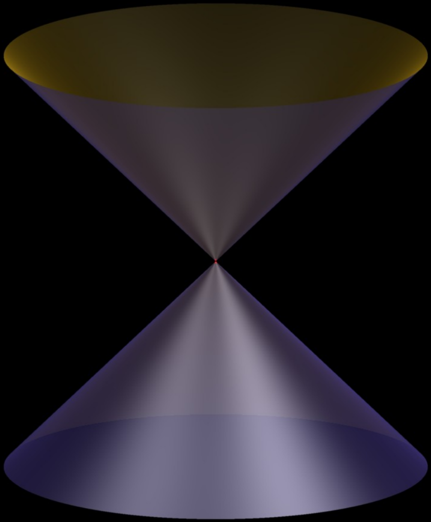


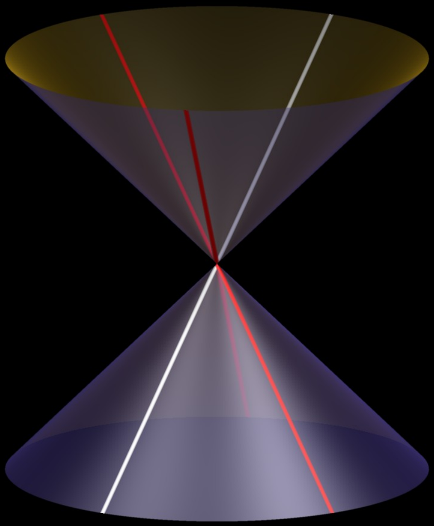
intersections

SINGULARITIES

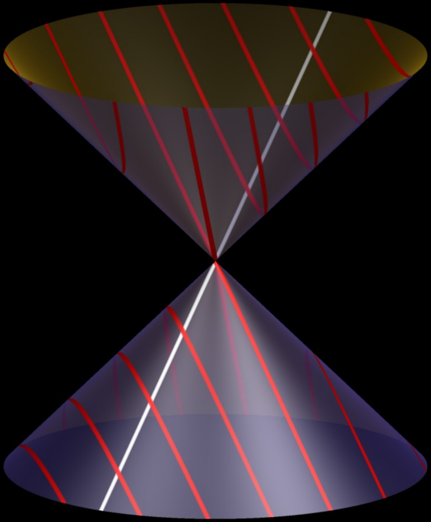
Singularities: cone



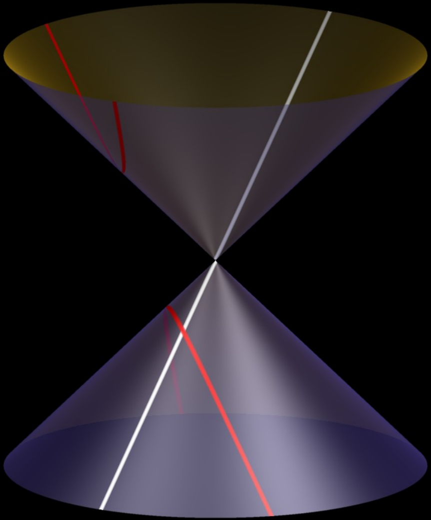
Singularities: 2 lines vs. 1 line



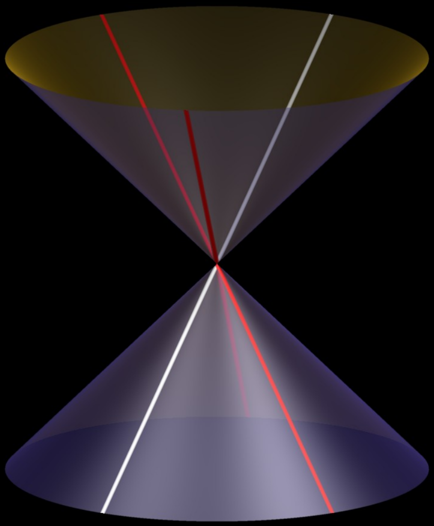
Singularities: deformation



Singularities: $\sim$2 lines



Singularities: 2 lines vs. 1 line



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Non-singular case

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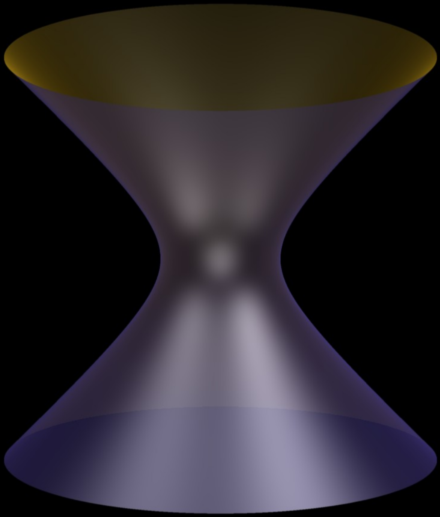
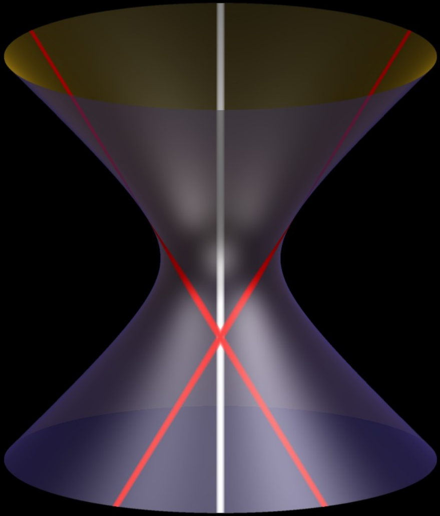
smoothing



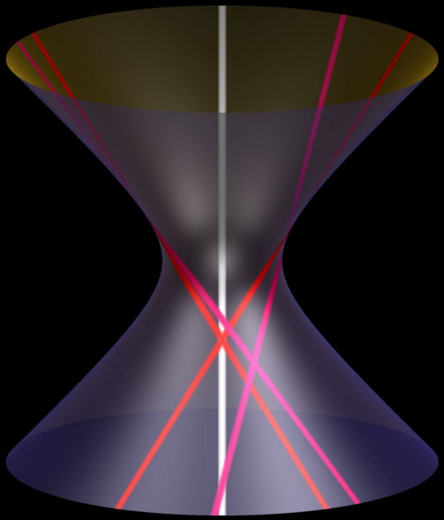
Non-singular case

2 lines vs. 1 line



Non-singular case

deformation



Non-singular case

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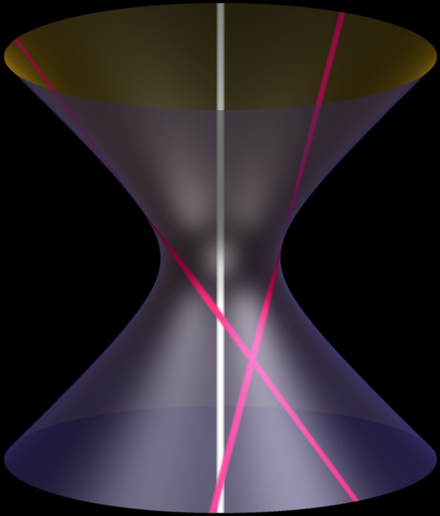
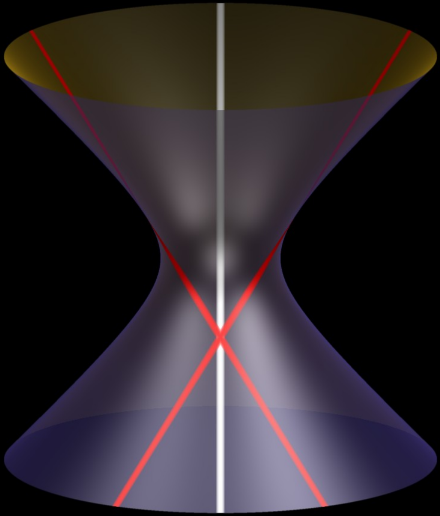


Non-singular case

2 lines vs. 1 line



Research

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- Reformulation:

$$\left(\frac{a}{c}\right)^n + \left(\frac{b}{c}\right)^n = 1 \qquad n \geq 3$$

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- Question: If $f(x, y)$ is a polynomial in x, y of degree ≥ 3 with integer coefficients, does

$$f(x, y) = 0$$

  have no solution with x, y non-zero rational numbers?

Geometry

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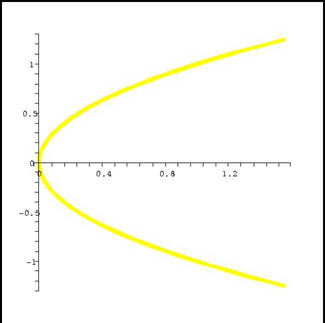

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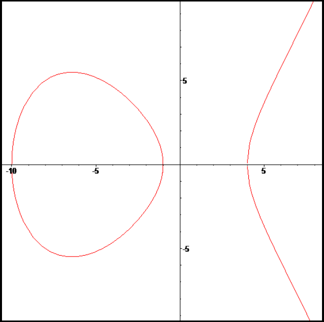

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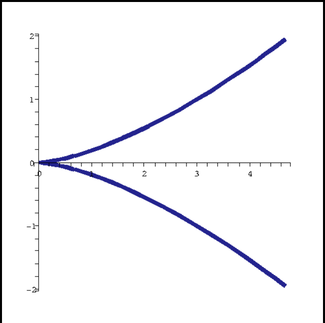

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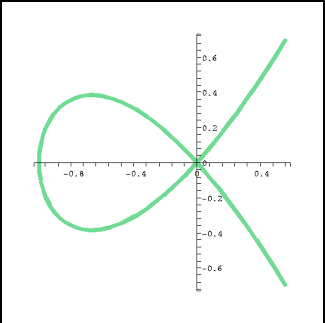

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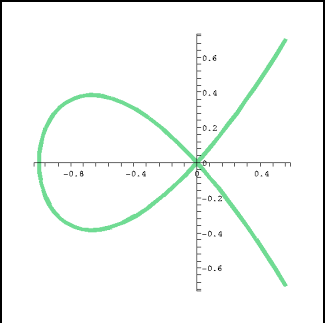

A solution with x, y rational numbers corresponds to a point on the curve with rational coordinates.

From Arithmetic to Geometry

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Are there solutions that can be expressed as polynomials of t? Let $x = 5t$ and $y = 2t^2$.

Mordell Conjecture

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- There is still something interesting to ask:
- Question: Is it true that for any given parametrization there are only finitely many solutions (in t)?
- This is known as Mordell's Conjecture and was confirmed by Manin in 1963.

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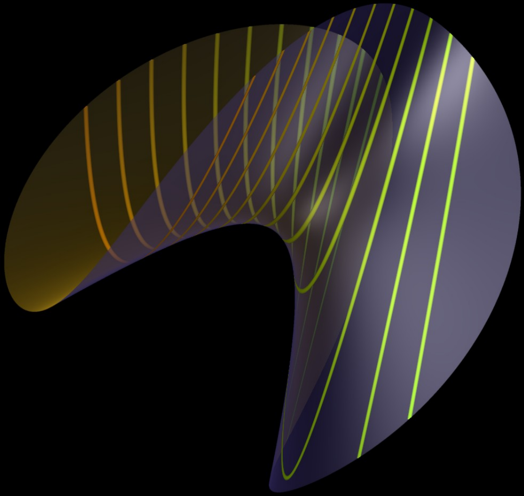
- In 1968 Parshin realized that this is related to another conjecture made by Shafarevich in 1962.
- The connection is somewhat tricky, but the point is that instead of looking for solutions (in t) the question focuses on studying the "total space" of the curves:
- As t varies, there is a curve C_t in the plane that is defined by the equation $f(x, y, t) = 0$. The union of these curves form a surface, which is "fibred over the t-line":

Shafarevich's Conjecture



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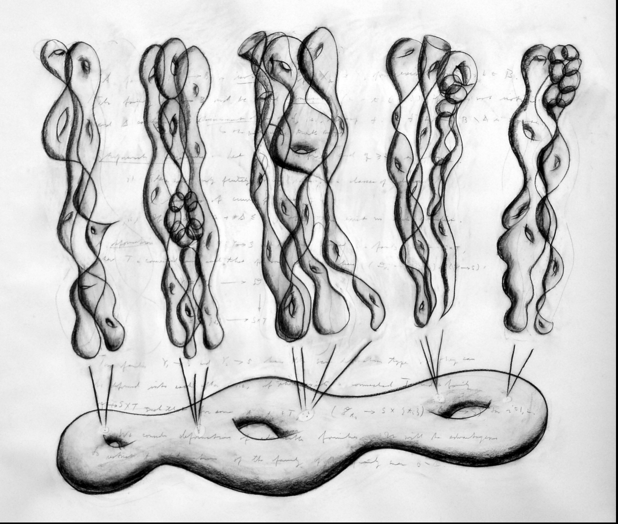

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- Parshin and Arakelov proved Shafarevich's Conjecture in 1968 – 1971.

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