ARE MINIMAL DEGREE RATIONAL CURVES DETERMINED BY THEIR TANGENT VECTORS?

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ABSTRACT. Let X be a projective variety which is covered by rational curves, for instance a Fano manifold over the complex numbers. In this setup, characterization and classification problems lead to the natural question: "Given two points on X, how many minimal degree rational curve are there which contain those points?". A recent answer to this question led to a number of new results in classification theory. As an infinitesimal analogue, we ask "How many minimal degree rational curves exist which contain a prescribed tangent vector?"

In this paper, we give sufficient conditions which guarantee that every tangent vector at a general point of X is contained in at most one rational curve of minimal degree. As an immediate application, we obtain irreducibility criteria for the space of minimal rational curves.

CONTENTS

1. INTRODUCTION

The study of rational curves of minimal degree has proven to be a very useful tool in Fano geometry. The spectrum of application covers diverse topics such as deformation rigidity, stability of the tangent sheaf, classification problems or the existence of non-trivial finite morphisms between Fano manifolds; see [Hwa01] for an overview.

In this paper we will consider the situation where X is a projective variety, which is covered by rational curves, e.g. a Fano manifold over $\mathbb{C}$. An example of that is $\mathbb{P}^n$, which is covered by lines. The key point of many applications of minimal degree rational curves is showing that the curves in question are similar to lines in certain respects. For instance, one may ask:

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Question 1.1. *Under what conditions does there exist a unique minimal degree rational curve containing two given points?*

This question found a sharp answer in [Keb02a], see [CMSB00] and [Keb02b] for a number of applications. The argument used there is based on a criterion of Miyaoka, who was the first to observe that if the answer to the question is “No”, then a lot of minimal degree curves are singular. We refer to [Kol96, Prop. V.3.7.5] for a precise statement.

As an infinitesimal analogue of this question one may ask the following:

Question 1.2. *Are there natural conditions that guarantee that a minimal degree rational curve is uniquely determined by a tangent vector?*

Although a definite answer to the latter question would be as interesting as one to the former, it seems that Question 1.2 has hardly been studied before. This paper is a first attempt in that direction. We give a criterion which parallels Miyaoka’s approach.

Theorem 1.3. *Let X be a projective variety over an algebraically closed field k and $H \subset \mathrm{RatCurves}^{\mathrm{n}}(X)$ a proper, covering family of rational curves such that none of the associated curves has a cuspidal singularity. If $\mathrm{char}(k) \neq 0$, assume additionally that there exists an ample line bundle $L \in \mathrm{Pic}(X)$ such that for every $\ell \in H$ the intersection number $L.\ell$ of L is coprime to $\mathrm{char}(k)$.*

Then, if $x \in X$ is a general point, all curves associated with the closed subfamily

$$H_x := \{\ell \in H \mid x \in \ell\} \subset H$$

are smooth at x and no two of them share a common tangent direction at x.

Remark 1.3.1. In Theorem 1.3 we do not assume that H is irreducible or connected. That will later be important for the applications.

Remark 1.3.2. We refer the reader to Chapter 3.3.1 for a brief review of the space $\mathrm{RatCurves}^{\mathrm{n}}(X)$ of rational curves. The volume [Kol96] contains a thorough discussion.

If $H \subset \mathrm{RatCurves}^{\mathrm{n}}(X)$ is an irreducible component, it is known that H is proper if there exists a line bundle $L \in \mathrm{Pic}(X)$ that intersects a curve $\ell \in H$ with multiplicity $L.\ell = 1$.

For complex projective manifolds we give another result. To formulate the setup properly, pick an irreducible component $H \subset \mathrm{RatCurves}^{\mathrm{n}}(X)$ such that

(1) the rational curves associated with H dominate X,
(2) for a general point $x \in$, the closed subfamily H_x is proper.

Let $\tilde{U}$ be the universal family, which is a $\mathbb{P}^1$-bundle over H. The tangent map of the natural projection $\iota : \tilde{U} \to X$, restricted to the relative tangent sheaf $T_{\tilde{U}/H}$, gives rise to a rational map τ:

$$\begin{tikzcd}
 & \mathbb{P}(T_X^\vee) \arrow[d] \\
\tilde{U} \arrow[ur, dashed, bend left, "\tau"] \arrow[r, "\iota", "\text{evaluation}"'] \arrow[d, "\pi"', "\mathbb{P}^1\text{-bundle}"] & X \\
H &
\end{tikzcd}$$

It has been shown in [Keb02a] that τ is well-defined and finite over an open set of X. Examples of rationally connected manifolds, however, seem to suggest that the tangent map τ is generically injective for a large class of varieties. Our main result supports this claim.

Theorem 1.4. *Let X be a smooth projective variety over the field of complex numbers and let $H \subset \mathrm{RatCurves}^{\mathrm{n}}(X)$ be the union of irreducible components such that the subfamily H_x is proper for all points $x \in X$, outside a subvariety $S \subset X$ of codimension at least 2.*

Then τ is generically injective, unless the curves associated with the closed subfamily $H^{cusp} \subset H$ of cuspidal curves dominate X, and the subvariety

$$D := \{x \in X \mid \exists \ell \in H^{cusp} : \ell \text{ has a cuspidal singularity at } x\},$$

where curves have cuspidal singularities, has codimension 1.

Remark 1.4.1. It is known that the family H_x is proper for a general point $x \in X$ if H is a "maximal dominating family of rational curves of minimal degrees", i.e., if the degrees of the curves associated with H are minimal among all irreducible components of $\mathrm{RatCurves}^{\mathrm{n}}(X)$ which satisfy condition (1) from above.

The assumption that H_x is proper for all points outside a set of codimension 2, however, is restrictive.

The structure of the article is as follows. In Section 2 we discuss some basic facts about $\mathbb{P}^1$-bundles with an irreducible double section. This is elementary, but turns out to be important later. A central element of the proofs of 1.3 and 1.4 is the study of families of *dubbies*, that is, reducible curves that consist of touching rational curves. Section 3 contains the precise definition and relevant properties of dubbies. The actual proofs are included in Section 4.

Although we consider the main results to be interesting on their own, we also present several applications in Section 5.

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After the main part of this paper was written, J.-M. Hwang has informed us that, together with N. Mok, they have shown a statement similar to, but somewhat stronger than Theorem 1.4. Their unpublished proof uses entirely different methods. To the best of our knowledge, there is no other result similar to Theorem 1.3.

2. $\mathbb{P}^1$-BUNDLES WITH DOUBLE SECTIONS

This preliminary section discusses $\mathbb{P}^1$-bundles with an irreducible double section. Most results here are fairly elementary. We have, however, chosen to include detailed proofs for lack of a suitable reference.

Throughout the present section let $\lambda : \Lambda \to B$ be a $\mathbb{P}^1$-bundle over a normal variety B, i.e., a morphism whose scheme-theoretic fibers are all isomorphic to $\mathbb{P}^1$. Let $\sigma : B \to \Lambda$ be a section of λ, $\Sigma_{\mathrm{red}} = \sigma(B)_{\mathrm{red}} \subset \Lambda$, and let $\Sigma \subset \Lambda$ be the first infinitesimal neighborhood of Σ_{red} in Λ. That is, if Σ_{red} is defined by the sheaf of ideals $\mathcal{J} = \mathcal{O}_\Lambda(-\Sigma_{\mathrm{red}})$, then Σ is defined by the sheaf $\mathcal{J}^2$. Our aim is to relate properties of Λ with those of its subscheme Σ.

2.1. **The Picard group of the double section.** Recall from [Har77, III. Ex.4.6] that there exists a short exact sequence of sheaves of Abelian groups, sometimes called that "truncated exponential sequence" in the literature (eg. [BBI00, sect. 2])

$$0 \longrightarrow \underbrace{\mathcal{J}/\mathcal{J}^2}_{=N^\vee_{\Sigma_{\mathrm{red}}|\Lambda}} \xrightarrow{\alpha} \mathcal{O}^*_\Sigma \xrightarrow{\beta} \mathcal{O}^*_{\Sigma_{\mathrm{red}}} \longrightarrow 1. \tag{2.1}$$

Here $N^\vee_{\Sigma_{\mathrm{red}}|\Lambda}$ is the conormal bundle, β is the canonical restriction map and α is given by

$$\begin{array}{rccc} \alpha: & (\mathcal{J}/\mathcal{J}^2, +) & \to & \mathcal{O}^*_\Sigma \\ & f & \mapsto & 1+f. \end{array}$$

In our setup, where $\Sigma_{\mathrm{red}} \simeq B$ is a section, the truncated exponential sequence (2.1) is canonically split. Locally we can write the splitting as follows. Assume that we are given an affine open subset $U_\alpha \subset \Sigma$ and an invertible function $f_\alpha \in \mathcal{O}^*_\Sigma(U_\alpha)$. Then, after shrinking U_α, if needed, we will find a bundle coordinate y_a, identify

$$\mathcal{O}^*_\Sigma(U_\alpha) \simeq [\mathcal{O}_{\Sigma_{\mathrm{red}}}(U_\alpha) \otimes k[y_\alpha]/(y_\alpha^2)]^*$$

and write accordingly

$$f_\alpha = g_\alpha + h_\alpha \cdot y_\alpha$$

where $g_\alpha \in \mathcal{O}^*_{\Sigma_{\mathrm{red}}}(U_\alpha)$ and $h_\alpha \in \mathcal{O}_{\Sigma_{\mathrm{red}}}(U_\alpha)$. With this notation, the splitting of sequence (2.1) decomposes f_α as

$$f_\alpha = g_\alpha \cdot \underbrace{\left[1 + \frac{h_\alpha}{g_\alpha} \cdot y_\alpha\right]}_{\in \mathrm{Im}(\alpha_{U_\alpha})}$$

As a direct corollary to the splitting of (2.1) we obtain a canonical decomposition of the Picard group

$$\mathrm{Pic}(\Sigma) = \mathrm{Pic}(\Sigma_{\mathrm{red}}) \times H^1(\Sigma_{\mathrm{red}}, N^\vee_{\Sigma_{\mathrm{red}}|\Lambda}). \tag{2.2}$$

2.2. **The cohomology class of a line bundle.** Let $L \in \mathrm{Pic}(\Lambda)$ be a line bundle. Using the decomposition (2.2) from above, we can associate to L a class $c(L) \in H^1(\Sigma_{\mathrm{red}}, N^\vee_{\Sigma_{\mathrm{red}}|\Lambda})$. As this class will be important soon, we will now find a Čech-cocycle in $Z^1(U_\alpha, N^\vee_{\Sigma_{\mathrm{red}}|\Lambda})$ that represents $c(L)$.

To this end, find a suitable open affine cover U_α of Σ such that $L|_{U_\alpha}$ is trivial for all α and where bundle coordinates y_α exist. Let $f_\alpha \in L(U_\alpha)$ be a collection of nowhere vanishing sections which we write in local coordinates as $f_\alpha = g_\alpha + h_\alpha \cdot y_\alpha$. Using the U_α-coordinates on the intersection $U_\alpha \cap U_\beta$, the transition functions for the line bundle are thus written as

$$\frac{f_\alpha}{f_\beta} = \frac{g_\alpha + h_\alpha \cdot y_\alpha}{g_\beta + h_\beta \cdot y_\alpha} = \frac{g_\alpha}{g_\beta} \cdot \left[1 + \left(\frac{h_\alpha}{g_\alpha} - \frac{h_\beta}{g_\beta}\right) y_\alpha\right] \in \mathcal{O}^*_\Sigma(U_{\alpha\beta})$$

In other words, the class of $c(L) \in H^1(\Sigma_{\mathrm{red}}, N^\vee_{\Sigma_{\mathrm{red}}|\Lambda})$ is represented by the Čech cocycle

$$\left(\frac{h_\alpha}{g_\alpha} - \frac{h_\beta}{g_\beta}\right) y_\alpha \in Z^1(U_\alpha, N^\vee_{\Sigma_{\mathrm{red}}|\Lambda}) \tag{2.3}$$

2.3. **Vector bundle sequences associated to line bundles.** Consider the ideal sheaf sequence for $\Sigma_{\mathrm{red}} \subset \Sigma$.

$$0 \longrightarrow \mathcal{J}\big/\mathcal{J}^2 \longrightarrow \mathcal{O}_\Sigma \longrightarrow \mathcal{O}_{\Sigma_{\mathrm{red}}} \longrightarrow 0$$

Warning 2.1. It should be noted that Σ_{red} is not a Cartier-divisor in Σ since its ideal sheaf, $\mathcal{J}\big/\mathcal{J}^2 \simeq N^\vee_{\Sigma_{\mathrm{red}}|\Lambda}$ is *not* a locally free $\mathcal{O}_\Sigma$-module. Furthermore, the restriction of the ideal sheaf of Σ_{red} in Λ to Σ, $\mathcal{J} \otimes \mathcal{O}_\Sigma \simeq \mathcal{J}\big/\mathcal{J}^3$, is not isomorphic to the ideal sheaf of Σ_{red} in Σ, $\mathcal{J}\big/\mathcal{J}^2 \not\simeq \mathcal{J} \otimes \mathcal{O}_\Sigma$. In fact, $\mathcal{J} \otimes \mathcal{O}_\Sigma$ is not even a subsheaf of $\mathcal{O}_\Sigma$.

Construction 2.2. Let $L \in \mathrm{Pic}(\Sigma)$ be a line bundle. By abuse of notation, identify Σ_{red} with B and consider $L|_{\Sigma_{\mathrm{red}}}$ a line bundle on B. Then twist the above sequence with the locally free $\mathcal{O}_\Sigma$-module $L \otimes \lambda^*(L^\vee|_{\Sigma_{\mathrm{red}}})$, and obtain the following sequence of $\mathcal{O}_\Sigma$-modules,

$$0 \longrightarrow N^\vee_{\Sigma_{\mathrm{red}}|\Lambda} \longrightarrow L \otimes \lambda^*(L^\vee|_{\Sigma_{\mathrm{red}}}) \longrightarrow \mathcal{O}_{\Sigma_{\mathrm{red}}} \longrightarrow 0. \tag{2.4}$$

Finally, consider the push-forward to B:

$$0 \longrightarrow N^\vee_{\Sigma_{\mathrm{red}}|\Lambda} \longrightarrow \underbrace{\lambda_*(L) \otimes L^\vee|_{\Sigma_{\mathrm{red}}}}_{=:\mathcal{E}_L} \xrightarrow{A} \mathcal{O}_{\Sigma_{\mathrm{red}}} \longrightarrow 0. \tag{2.5}$$

We obtain a vector bundle $\mathcal{E}_L$ of rank two on B which is presented as an extension of two line bundles. The surjective map $\mathcal{E}_L \to \mathcal{O}_B$ induces a section $\sigma_L : B \to \mathbb{P}(\mathcal{E})$. We will use this notation later and also extend it to line bundles, $L \in \mathrm{Pic}(\Lambda)$, by $\mathcal{E}_L := \mathcal{E}_{L|_\Sigma}$ and $\sigma_L = \sigma_{L|_\Sigma}$. Observe that $(\mathbb{P}(\mathcal{E}_L), \sigma_L)$ depends on L only up to a twist by a line bundle pulled back from B. I.e., for $M \in \mathrm{Pic}(B)$, $\mathcal{E}_{L\otimes\lambda^* M} \simeq \mathcal{E}_L$ and $\sigma_{L\otimes\lambda^* M} = \sigma_L$.

Much of our further argumentation is based on the following observation.

Proposition 2.3. *Let $L \in \mathrm{Pic}(\Sigma)$ be a line bundle and $c(L) \in H^1(\Sigma_{\mathrm{red}}, N^\vee_{\Sigma_{\mathrm{red}}|\Lambda})$ the class defined above. Then $c(L)$ coincides with the extension class*

$$e(L) \in \mathrm{Ext}^1(\mathcal{O}_{\Sigma_{\mathrm{red}}}, N^\vee_{\Sigma_{\mathrm{red}}|\Lambda}) = H^1(\Sigma_{\mathrm{red}}, N^\vee_{\Sigma_{\mathrm{red}}|\Lambda})$$

of the vector bundle sequence (2.5)*. In particular, the map*

$$\begin{array}{cccc} e: & (\mathrm{Pic}(\Sigma), \otimes) & \to & (H^1(\Sigma_{\mathrm{red}}, N^\vee_{\Sigma_{\mathrm{red}}|\Lambda}), +) \\ & L & \mapsto & \textit{extension class of sequence } (2.5) \end{array}$$

is a homomorphism of groups.

Proof. The proof relies on an explicit calculation in Čech cohomology. We will choose a sufficiently fine cover U_α of Σ_{red} and produce a Čech cocycle in $Z^1(U_\alpha, N^\vee_{\Sigma_{\mathrm{red}}|\Lambda})$ that represents the extension class $e(L)$. It will turn out that this cocycle equals the one that we have calculated in (2.3) above for $c(L)$.

We keep the notation from above and let $f_\alpha \in L(U_\alpha)$ be a collection of nowhere-vanishing sections of L. Such sections can be naturally seen to give local splittings of the sequences (2.4) and (2.5). Explicitly, if we write $f_\alpha = g_\alpha + h_\alpha \cdot y_\alpha$, then

$$\frac{f_\alpha}{g_\alpha} = 1 + \frac{h_\alpha}{g_\alpha} \cdot y_\alpha \in (L \otimes L^\vee|_{\Sigma_{\mathrm{red}}})(U_\alpha)$$

are nowhere-vanishing sections of $L \otimes L^\vee|_{\Sigma_{\mathrm{red}}}$ and the splitting takes the form

$$\begin{array}{cccc} s_\alpha: & \mathcal{O}_{\Sigma_{\mathrm{red}}}(U_\alpha) & \to & (L \otimes L^\vee|_{\Sigma_{\mathrm{red}}})(U_\alpha) \\ & 1 & \mapsto & 1 + \frac{h_\alpha}{g_\alpha} \cdot y_\alpha \end{array}$$

By construction of Ext^1, we obtain the extension class as the homology class represented by the Čech cocycle

$$s_\alpha(1) - s_\beta(1) \in \ker(A)(U_{\alpha\beta}) \simeq N^\vee_{\Sigma_{\mathrm{red}}|\Lambda}(U_{\alpha\beta})$$

This difference is given by the following section in $N^\vee_{\Sigma_{\mathrm{red}}|\Lambda}(U_{\alpha\beta})$ which yields the required cocycle.

$$\left(1 + \frac{h_\alpha}{g_\alpha} \cdot y_\alpha\right) - \left(1 + \frac{h_\beta}{g_\beta} \cdot y_\alpha\right) = \left(\frac{h_\alpha}{g_\alpha} - \frac{h_\beta}{g_\beta}\right) y_\alpha \in Z^1(U_\alpha, N^\vee_{\Sigma_{\mathrm{red}}|\Lambda})$$

That, however, is the same cocycle which we have obtained above in formula (2.3) for the class $c(L)$. The proof of Proposition 2.3 is therefore finished. $\square$

2.4. **The reconstruction of the $\mathbb{P}^1$-bundle from a double section.** It is a remarkable fact that the restriction of an ample line bundle $L \in \mathrm{Pic}(\Lambda)$ to a double section carries enough information so that the whole $\mathbb{P}^1$-bundle Λ can be reconstructed. The proof is little more than a straightforward application of Proposition 2.3. We are grateful to Ivo Radloff who showed us how to use extension classes to simplify our original proof.

Notation 2.4. Let (Λ, σ) and (Λ', σ') be two $\mathbb{P}^1$-bundles with sections over B. We say that (Λ, σ) and (Λ', σ') are *isomorphic pairs (over B)* if there exists a morphism $\gamma : \Lambda \to \Lambda'$, an *isomorphism of pairs*, such that γ is a B-isomorphism of $\mathbb{P}^1$-bundles and $\gamma \circ \sigma = \sigma'$. Sometimes we will refer to these pairs by the image of the section: $(\Lambda, \sigma(B))$, in which case the meaning of *isomorphic pairs* should be clear.

Theorem 2.5. *Given a line bundle $L \in \mathrm{Pic}(\Lambda)$, which is not the pull-back of a line bundle on B, let $\mathcal{E}_L$ and σ_L be as in 2.2. Consider the relative degree $d \in \mathbb{Z} \setminus \{0\}$ of L, i.e., the intersection number with fibers of λ. If d is coprime to $\mathrm{char}(k)$, then (Λ, σ) and $(\mathbb{P}(\mathcal{E}_L), \sigma_L)$ are isomorphic pairs over B.*

Proof. Let $H := \mathcal{O}_\Lambda(\Sigma_{\mathrm{red}}) = \mathcal{J}^\vee$. Then $\Lambda \simeq \mathbb{P}(\lambda_* H)$ and $\sigma : B \to \Lambda$ is the section associated to the surjection, $\lambda_* H \to \lambda_*(H|_{\Sigma_{\mathrm{red}}})$.

First we would like to prove that $\lambda_* H \simeq \lambda_*(H|_\Sigma)$. Indeed, consider the sequence,

$$0 \longrightarrow H \otimes \mathcal{J}^2 \simeq \mathcal{J} \longrightarrow H \longrightarrow H|_\Sigma \longrightarrow 0.$$

We need to prove that $\lambda_* \mathcal{J} \simeq R^1\lambda_* \mathcal{J} \simeq 0$. However, that follows from considering the push-forward of the sequence,

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_\Lambda \longrightarrow \mathcal{O}_{\Sigma_{\mathrm{red}}} \longrightarrow 0,$$

since $\lambda_* \mathcal{O}_\Lambda \simeq \lambda_* \mathcal{O}_{\Sigma_{\mathrm{red}}} \simeq \mathcal{O}_B$, and $R^1\lambda_* \mathcal{O}_\Lambda \simeq 0$.

This implies the statement for $L = H$, that is, we obtain that (Λ, σ) and $(\mathbb{P}(\mathcal{E}_H), \sigma_H)$ are isomorphic pairs over B (cf. [Har77, II.7.9]).

In order to finish the proof, we are going to prove that $(\mathbb{P}(\mathcal{E}_H), \sigma_H)$ and $(\mathbb{P}(\mathcal{E}_L), \sigma_L)$ are isomorphic pairs over B for any $L \in \mathrm{Pic}(\Lambda)$. In fact, it suffices to show that the extension classes of the following sequences are the same up to a non-zero scalar multiple.

$$\begin{aligned} 0 \longrightarrow N^\vee_{\Sigma_{\mathrm{red}}|\Lambda} \longrightarrow \lambda_*(H|_\Sigma) \otimes H^\vee|_{\Sigma_{\mathrm{red}}} \longrightarrow \mathcal{O}_{\Sigma_{\mathrm{red}}} \longrightarrow 0 \\ 0 \longrightarrow N^\vee_{\Sigma_{\mathrm{red}}|\Lambda} \longrightarrow \lambda_*(L|_\Sigma) \otimes L^\vee|_{\Sigma_{\mathrm{red}}} \longrightarrow \mathcal{O}_{\Sigma_{\mathrm{red}}} \longrightarrow 0. \end{aligned} \tag{2.6}$$

Recall that $\mathrm{Pic}(\Lambda) = \mathbb{Z} \times \mathrm{Pic}(B)$ so that we can write $L \in H^{\otimes d} \otimes \lambda^* M$ for an appropriate $M \in \mathrm{Pic}(B)$. By Proposition 2.3 this implies that the extension classes of the sequences (2.6) are given by $c\left(H|_\Sigma\right)$ and $c\left(H|_\Sigma^{\otimes d}\right) = d \cdot c\left(H|_\Sigma\right)$. In particular, they differ only by the non-zero factor $d \in k$. $\square$

Warning 2.6. The construction of the vector bundle $\mathcal{E}_L$ and Proposition 2.3 use only the restriction $L|_\Sigma$. It may thus appear that Theorem 2.5 could be true without the assumption that $L \in \mathrm{Pic}(\Lambda)$ and that one could allow arbitrary line bundles $L \in \mathrm{Pic}(\Sigma)$ instead. That, however, is wrong and counterexamples do exist. Note that the proof of Theorem 2.5 uses the fact that L is contained in $\mathbb{Z} \times \mathrm{Pic}(B)$ which is not true in general if $L \in \mathrm{Pic}(\Sigma)$ is arbitrary.

The assumption that d be coprime to $\mathrm{char}(k)$ is actually necessary in Theorem 2.5, as shown by the following simple corollary of Proposition 2.3 and of the proof of Theorem 2.5.

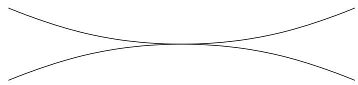
Corollary 2.7. *Using the same notation as in Theorem 2.5, assume that d is divisible by $\mathrm{char}(k)$. Then*

$$\lambda_*(L|_\Sigma) \otimes L^\vee|_{\Sigma_{\mathrm{red}}} \simeq N^\vee_{\Sigma_{\mathrm{red}}|\Lambda} \oplus \mathcal{O}_{\Sigma_{\mathrm{red}}}.$$

3. DUBBIES

Throughout the proofs of Theorems 1.3 and 1.4, which we give in Sections 4.1 and 4.2 below, we will assume that X contains pairs of minimal rational curves which intersect tangentially in at least one point. A detailed study of these pairs and their parameter spaces will be given in the present chapter. The simplest configuration is the following:

Definition 3.1. *A* dubby *is a reduced, reducible curve, isomorphic to the union of a line and a smooth conic in $\mathbb{P}^2$ intersecting tangentially in a single point.*



Remark 3.1.1. The definition may suggest at first glance that one component of a dubby is special in that it has a higher degree than the other. We remark that this is not so. A dubby does not come with a natural polarization. In fact, there exists an involution in the automorphism group that swaps the irreducible components.

Later we will need the following estimate for the dimension of the space of global sections of a line bundle on a dubby. Let $\ell = \ell_1 \cup \ell_2$ be a dubby and $L \in \mathrm{Pic}(\ell)$ a line bundle. We say that L has type (d_1, d_2) if the restrictions of L to the irreducible components ℓ_1 and ℓ_2 have degree d_1 and d_2, respectively.

Lemma 3.2. *Let ℓ be a dubby and $L \in \mathrm{Pic}(\ell)$ a line bundle of type (d_1, d_2). Then $h^0(\ell, L) \geq d_1 + d_2$.*

Proof. By assumption, we have that $L|_{\ell_i} \simeq \mathcal{O}_{\mathbb{P}^1}(d_i)$. Let $\ell_1.\ell_2$ be the scheme theoretic intersection of ℓ_1 and ℓ_2, $\iota^i : \ell_i \to \ell$ the natural embedding, and $L_i = \iota^i_*(L|_{\ell_i})$ for $i = 1, 2$. Then one has the following short exact sequence:

$$0 \to L \to L_1 \oplus L_2 \to \mathcal{O}_{\ell_1.\ell_2} \to 0.$$

This implies that $h^0(\ell, L) \geq \chi(L) = \chi(L_1) + \chi(L_2) - \chi(\mathcal{O}_{\ell_1.\ell_2}) = d_1 + d_2$. □

3.1. **The identification of the components of a dubby.** To illustrate the main observation about dubbies, let us consider a very simple setup first: let $L \in \mathrm{Pic}(X)$ be an ample line bundle, and assume that $\ell = \ell_1 \cup \ell_2 \subset X$ is a dubby where both components are members of the same connected family H of minimal rational curves. In particular, $L|_\ell$ will be of type (d, d), where $d > 0$. Remarkably, the line bundle L induces a canonical identification of the two components ℓ_1 and ℓ_2, at least when d is coprime to the characteristic of the base field k. Over the field of complex numbers, the idea of construction is the following: Fix a trivialization $t : L|_V \to \mathcal{O}_V$ of L on an open neighborhood V of the intersection point $\{z\} = \ell_1 \cap \ell_2$. Given a point $x \in \ell_1 \setminus \ell_2$, let $\sigma_1 \in H^0(\ell_1, L|_{\ell_1})$ be a non-zero section that vanishes at x with multiplicity d. Then there exists a *unique* section $\sigma_2 \in H^0(\ell_2, L|_{\ell_2})$ with the following properties:

(1) The section σ_2 vanishes at exactly one point $y \in \ell_2$.
(2) The sections σ_1 and σ_2 agree on the intersection of the components:

$$\sigma_1(z) = \sigma_2(z)$$

(3) The differentials of σ_1 and σ_2 agree at z:

$$\vec{v}(t \circ \sigma_1) = \vec{v}(t \circ \sigma_2)$$

for all non-vanishing tangent vectors $\vec{v} \in T_{\ell_1} \cap T_{\ell_2}$.

The map that associates x to y gives the identification of the components and does not depend on the choice of t.

In the following section 3.2, we will give a construction of the identification morphism which also works in the relative setup, for bundles of type (d_1, d_2) where $d_1 \neq d_2$, and in arbitrary characteristic.

3.2. **Bundles of dubbies.** For the proof of the main theorems we will need to consider bundles of dubbies, i.e., morphisms where each scheme-theoretic fiber is isomorphic to a dubby. The following Proposition shows how to identify the components of such bundles.

Proposition 3.3. *Let $\lambda : \Lambda \to B$ be a projective family of dubbies over a normal base B and assume that Λ is not irreducible. Then it has exactly two irreducible components Λ_1 and Λ_2, both $\mathbb{P}^1$-bundles over B. Assume further that there exists a line bundle $L \in \operatorname{Pic}(\Lambda)$ whose restriction to a λ-fiber has type (m, n), where m and n are non-zero and relatively prime to $\operatorname{char}(k)$.*

If $\Sigma_{\mathrm{red}} \subset \Lambda_1 \cap \Lambda_2$ denotes the reduced intersection, then Σ_{red} is a section over B, and the pairs $(\Lambda_1, \Sigma_{\mathrm{red}})$ and $(\Lambda_2, \Sigma_{\mathrm{red}})$ are isomorphic over B.

Note that the isomorphism given in Proposition 3.3 is not canonical and may not respect the line bundle L.

Proof of Proposition 3.3. The map λ is flat because all its scheme-theoretic fibers are isomorphic. Let $\Lambda_1 \subset \Lambda$ be one of the irreducible components. It is easy to see that if $x \in \Lambda_1$ is a general point, then Λ_1 contains the (unique) irreducible component of $\ell_{\lambda(x)} := \lambda^{-1}\lambda(x)$ that contains x. Since λ is proper and flat, $\lambda(\Lambda_1) = B$. Hence Λ_1 contains one of the irreducible components of ℓ_b for all $b \in B$. Repeating the same argument with another irreducible component, Λ_2, one finds that it also contains one of the irreducible components of ℓ_b for all $b \in B$. However, they cannot contain the same irreducible component for any $b \in B$: In fact, if they contained the same component of ℓ_b for infinitely many points $b \in B$, then they would agree. On the other hand, if they contained the same component of ℓ_b for finitely many points $b \in B$, then Λ would have an irreducible component that does not dominate B. This, however, would contradict the flatness of λ. Hence $\Lambda_1 \cup \Lambda_2 = \Lambda$. They are both $\mathbb{P}^1$-bundles over B by [Kol96, Thm. II.2.8.1].

Let $\Sigma := \Lambda_1 \cap \Lambda_2$ be the scheme-theoretic intersection. Since Λ is a bundle of dubbies and B is normal, it is clear that its reduction, Σ_{red} is a section, and that Σ is its first infinitesimal neighborhood in either Λ_1 or Λ_2. In this setup, the isomorphism of pairs is given by Theorem 2.5. □

3.3. **The space of dubbies.** In addition to the space of rational curves, which we use throughout, it is also useful to have a parameter space for dubbies. For the convenience of the reader, we will first recall the construction of the former space very briefly. The reader is referred to [Kol96, chapt. II.1] for a thorough treatment.

3.3.1. *The space of rational curves.* Recall that there exists a scheme $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$ whose geometric points correspond to morphisms $\mathbb{P}^1 \to X$ that are birational onto their images. Furthermore, there exists an "evaluation morphism": $\mu : \mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X) \times \mathbb{P}^1 \to X$. The group PGL_2 acts on the normalization $\mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X)$, and the geometric quotient exists. More precisely, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X) \times \mathbb{P}^1 \arrow[r, "U"] \arrow[d] \arrow[rr, bend left, "\mu"] & \mathrm{Univ}^{\mathrm{rc}}(X) \arrow[r, "\iota"] \arrow[d, "\pi"'] & X \\
\mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X) \arrow[r, "u"] & \mathrm{RatCurves}^{\mathrm{n}}(X) &
\end{tikzcd}$$

where u and U are principal PGL_2 bundles, π is a $\mathbb{P}^1$-bundle and the restriction of the "evaluation morphism" ι to any fiber of π is a morphism which is birational onto its image. The quotient space $\mathrm{RatCurves}^{\mathrm{n}}(X)$ is then the parameter space of rational curves on X. The letter "n" in $\mathrm{RatCurves}^{\mathrm{n}}$ may be a little confusing. It has nothing to do with the dimension of X and it's not a power. It serves as a reminder that the parameter space is the normalization of a suitable quasiprojective subset of the Chow variety.

It may perhaps look tempting to define a space of dubbies in a similar manner, as a quotient of the associated Hom-scheme. However, since geometric invariant theory becomes somewhat awkward for group actions on non-normal varieties, we have chosen another, elementary but somewhat lengthier approach. The space of dubbies will be constructed as a quasi-projective subvariety of the space of ordered pairs of pointed rational curves, and the universal family of dubbies will be constructed directly.

3.3.2. *Pointed rational curves.* It is easy to see that $\mathrm{RC}_\bullet(X) = \mathrm{Univ}^{\mathrm{rc}}(X)$ naturally parameterizes pointed rational curves on X and the pull-back of the universal family

$$\mathrm{Univ}^{\mathrm{rc}}_\bullet(X) = \mathrm{RC}_\bullet(X) \times_{\mathrm{RatCurves}^{\mathrm{n}}(X)} \mathrm{Univ}^{\mathrm{rc}}(X)$$

is the universal family of pointed rational curves over $\mathrm{RC}_\bullet(X)$. The identification morphism $\mathrm{RC}_\bullet(X) \to \mathrm{Univ}^{\mathrm{rc}}(X)$ and the identity map of $\mathrm{RC}_\bullet(X)$ gives a section of this universal family:

$$\begin{tikzcd}
\mathrm{Univ}^{\mathrm{rc}}_\bullet(X) \arrow[r] \arrow[d] & \mathrm{Univ}^{\mathrm{rc}}(X) \arrow[d] \\
\mathrm{RC}_\bullet(X) \arrow[r, "\pi"'] \arrow[ur, "\simeq"'] \arrow[u, bend left, dotted, "\eta"] & \mathrm{RatCurves}^{\mathrm{n}}(X)
\end{tikzcd}$$

3.3.3. *Ordered pairs of pointed rational curves.* The product $\mathrm{RC}^2_\bullet(X) := \mathrm{RC}_\bullet(X) \times \mathrm{RC}_\bullet(X)$ naturally parameterizes pairs of pointed rational curves. We denote the projections to the two factors by $\rho_i : \mathrm{RC}^2_\bullet(X) \to \mathrm{RC}_\bullet(X)$ for $i = 1, 2$. Then the universal family will be given as the disjoint union

$$\mathrm{Univ}^{\mathrm{rc},2}_\bullet(X) = \left(\mathrm{RC}^2_\bullet(X) \times_{\rho_1} \mathrm{Univ}^{\mathrm{rc}}_\bullet(X)\right) \cup \left(\mathrm{RC}^2_\bullet(X) \times_{\rho_2} \mathrm{Univ}^{\mathrm{rc}}_\bullet(X)\right).$$

The two copies of the section $\eta : \mathrm{RC}_\bullet(X) \to \mathrm{Univ}^{\mathrm{rc}}_\bullet(X)$ induce two sections of this family, one for each component of the union:

$$\begin{tikzcd}
 & & \mathrm{Univ}^{\mathrm{rc},2}_\bullet(X) \arrow[dll] \arrow[drr] \arrow[dd, "\tilde p"] & & \\
\mathrm{Univ}^{\mathrm{rc}}_\bullet(X) \arrow[dd] & & & & \mathrm{Univ}^{\mathrm{rc}}_\bullet(X) \arrow[dd] \\
 & & \mathrm{RC}^2_\bullet(X) \arrow[uu, dotted, bend left, "\sigma_1"] \arrow[uu, dotted, bend right, "\sigma_2"'] \arrow[dll, "\rho_1"'] \arrow[drr, "\rho_2"] & & \\
\mathrm{RC}_\bullet(X) \arrow[uu, dotted, bend left, "\eta"] & & & & \mathrm{RC}_\bullet(X) \arrow[uu, dotted, bend right, "\eta"']
\end{tikzcd}$$

3.3.4. *The space of dubbies.* Consider the evaluation morphism $\iota^2 : \mathrm{Univ}^{\mathrm{rc},2}_\bullet(X) \to X$. The associated tangent map $T\iota^2$ restricted to the relative tangent sheaf $T_{\mathrm{Univ}^{\mathrm{rc},2}_\bullet(X)/\,\mathrm{RC}^2_\bullet(X)}$ gives rise to a rational map

$$\tau^{\mathrm{rc},2} : \mathrm{Univ}^{\mathrm{rc},2}_\bullet(X) \dashrightarrow \mathbb{P}(T_X^\vee).$$

We define a quasiprojective variety, the space of dubbies,

$$\begin{aligned}\mathrm{Dubbies}^{\mathrm{n}}(X) := \text{normalization of } \{\ell \in \mathrm{RC}^2_\bullet(X) \,|\, \tau^{\mathrm{rc},2} \text{ is defined at } \sigma_1(\ell) \\ \text{and at } \sigma_2(\ell), \text{ and } \tau^{\mathrm{rc},2}(\sigma_1(\ell)) = \tau^{\mathrm{rc},2}(\sigma_2(\ell))\}.\end{aligned}$$

We will often consider pairs of curves such that both components come from the same family $H \subset \mathrm{RatCurves}^{\mathrm{n}}(X)$. For this reason we define $\pi_2 : \mathrm{Dubbies}^{\mathrm{n}}(X) \to \mathrm{RatCurves}^{\mathrm{n}}(X) \times \mathrm{RatCurves}^{\mathrm{n}}(X)$, the natural forgetful projection morphism, and

$$\mathrm{Dubbies}^{\mathrm{n}}(X)|_H := \mathrm{Dubbies}^{\mathrm{n}}(X) \cap \pi_2^{-1}(H \times H).$$

Proposition 3.4. *Assume that $H \subset \mathrm{RatCurves}^{\mathrm{n}}(X)$ is a proper family of immersed curves. Then $\mathrm{Dubbies}^{\mathrm{n}}(X)|_H$ is also proper.*

Proof. Since the tangent map, $\tau^{\mathrm{rc},2}$, is well-defined at $\sigma_1(\ell)$ and $\sigma_2(\ell)$ for every $\ell \in \mathrm{RC}^2_\bullet(X) \cap \pi_2^{-1}(H \times H)$,

$$\mathrm{Dubbies}^{\mathrm{n}}(X) = \text{normalization of } \{\ell \in \mathrm{RC}^2_\bullet(X) \,|\, \tau^{\mathrm{rc},2}(\sigma_1(\ell)) = \tau^{\mathrm{rc},2}(\sigma_2(\ell))\},$$

which is clearly a closed subvariety of the proper variety $\pi_2^{-1}(H \times H)$. □

The next statement follows immediately from the construction and from the universal property of $\mathrm{RatCurves}^{\mathrm{n}}(X)$.

Proposition 3.5. *Let ℓ_1 and $\ell_2 \subset X$ be rational curves with normalizations*

$$\eta_i : \mathbb{P}^1 \simeq \tilde\ell_i \to \ell_i \subset X.$$

If $T\eta_i$ have rank 1 at the point $[0:1] \in \mathbb{P}^1$ for $i = 1, 2$, and if the images of the tangent morphisms agree,

$$\mathrm{Image}(T\eta_1|_{[0:1]}) = \mathrm{Image}(T\eta_2|_{[0:1]}) \subset T_X,$$

then there exists a point $\ell \in \mathrm{Dubbies}^{\mathrm{n}}(X)$ such that $\tilde p^{-1}(\ell) = \tilde\ell_1 \cup \tilde\ell_2$.

If $H \subset \mathrm{RatCurves}^{\mathrm{n}}(X)$ is a subfamily, and both ℓ_i correspond to points of H, then we can find such an ℓ in $\mathrm{Dubbies}^{\mathrm{n}}(X)|_H$. □

Remark 3.5.1. Since $\mathrm{RC}^2_\bullet(X)$ is the space of ordered pairs of curves, the space $\mathrm{Dubbies}^{\mathrm{n}}(X)$ is really the space of 'ordered dubbies'. In other words, for each pair of rational curves with tangential intersection, there are at least two points of $\mathrm{Dubbies}^{\mathrm{n}}(X)$ representing it.

3.3.5. *The universal family of dubbies.* In order to show that $\mathrm{Dubbies}^{\mathrm{n}}(X)$ is a space of dubbies indeed, we need to construct a universal family, which is a bundle of dubbies in the sense of section 3.2. To this end, we will factor the universal evaluation morphism via a reducible family of dubbies.

Proposition 3.6. *The evaluation morphism,*

$$\iota : \underbrace{\mathrm{Univ}^{\mathrm{rc},2}_\bullet(X) \times_{\mathrm{RC}^2_\bullet(X)} \mathrm{Dubbies}^{\mathrm{n}}(X)}_{=:\tilde U,\ \textit{decomposes as}\ \tilde U_1 \cup \tilde U_2} \longrightarrow U \subset X \times \mathrm{Dubbies}^{\mathrm{n}}(X),$$

factors as follows.

$$\begin{tikzcd}
\tilde U \arrow[r, "\alpha"] \arrow[rr, bend left, "\iota"] \arrow[dr, "\tilde p"] & \Lambda \arrow[r, "\beta"] \arrow[d, "\hat p", "\textit{bundle of dubbies}"'] & U \arrow[dl, "p"'] \\
\textit{two disjoint } \mathbb{P}^1\textit{-bundles } \tilde U_1 \cup \tilde U_2 & \mathrm{Dubbies}^{\mathrm{n}}(X) & \textit{bundle of two curves with complicated intersection}
\end{tikzcd} \tag{3.1}$$

For every irreducible component $D \subset \mathrm{Dubbies}^{\mathrm{n}}(X)$, the preimage $\Lambda_D := \hat p^{-1}(D)$ is reducible, and is a bundle of dubbies in the sense that for every closed point $b \in D$, the fiber $\hat p^{-1}(b)$ is isomorphic to a dubby.

Remark 3.6.1. If $\ell \in \mathrm{Dubbies}^{\mathrm{n}}(X)$ is any point, then the two corresponding curves in X intersect tangentially in one point, but may have very complicated intersection at that point and elsewhere. The factorization of the evaluation morphism should therefore be understood as a partial resolution of singularities, as shown in figure 3.1.

Proof. As a first step we will construct the space Λ. Because the evaluation ι is a finite, hence affine, morphism, it seems appropriate to construct a suitable subsheaf $\mathcal{A} \subset \iota_* \mathcal{O}_{\tilde U}$, which is a coherent sheaf of $\mathcal{O}_U$-modules and set $\beta : \Lambda = \mathbf{Spec}(\mathcal{A}) \to U$.

Let $\bar\sigma_1 \subset \tilde U_1$ and $\bar\sigma_2 \subset \tilde U_2$ be the images of the pullbacks of the canonical sections, σ_1 and σ_2, constructed in 3.3.3. In order to construct $\mathcal{A}$, we will need to find an identification of their first infinitesimal neighborhoods, $\tilde\sigma_1$ and $\tilde\sigma_2$. Since ι is separable, it follows directly from the construction that $\tilde\sigma_1$ and $\tilde\sigma_2$ map isomorphically onto their scheme-theoretic images $\iota(\tilde\sigma_1)$ and $\iota(\tilde\sigma_2)$. Again, by the definition of $\mathrm{Dubbies}^{\mathrm{n}}(X)$, these images agree: $\iota(\tilde\sigma_1) = \iota(\tilde\sigma_2)$ and we obtain the desired identification,

$$\gamma : \tilde\sigma_1 \to \tilde\sigma_2.$$

Let

$$i_1 : \tilde\sigma_1 \to \tilde U_1 \quad \text{and} \quad i_2 : \tilde\sigma_2 \to \tilde U_2$$

be the inclusion maps and consider the sheaf morphism

$$\varphi := \iota_*(i_1^\# - \gamma^\# \circ i_2^\#) : \iota_* \mathcal{O}_{\tilde U} \to \iota_* \mathcal{O}_{\tilde\sigma_1}.$$

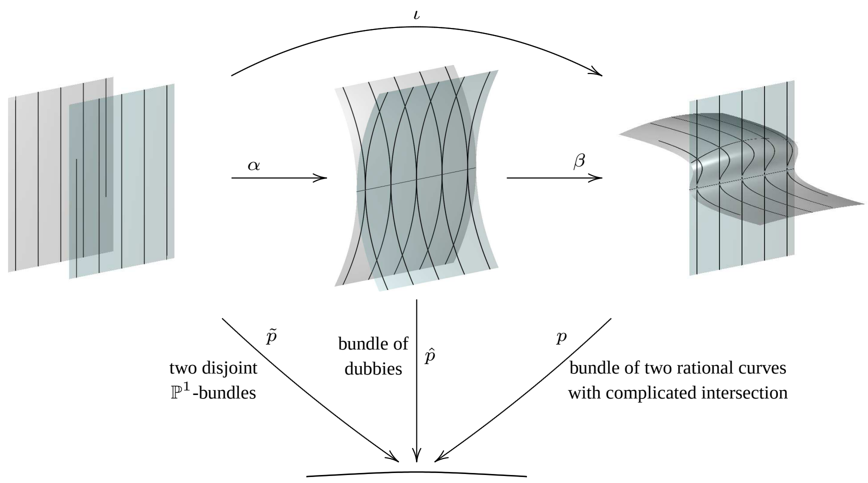


FIGURE 3.1. A partial resolution of singularities

The sheaf

$$\mathcal{A} := \ker(\varphi)$$

is thus a coherent sheaf of $\mathcal{O}_U$-modules. As it was planned above, define $\Lambda :=$ **Spec**$(\mathcal{A})$. The existence of the morphisms α and β and that $\iota = \beta \circ \alpha$ follows from the construction. It remains to show that Λ is a bundle of dubbies. Let $\ell \in \mathrm{Dubbies}^{\mathrm{n}}(X)$ be a closed point. Replacing $\mathrm{Dubbies}^{\mathrm{n}}(X)$ with a neighborhood of ℓ and passing to a finite, unbranched cover if necessary, and by abuse of notation still denoting it by $\mathrm{Dubbies}^{\mathrm{n}}(X)$, we can assume that

(1) the variety $\mathrm{Dubbies}^{\mathrm{n}}(X)$ is affine, say $\mathrm{Dubbies}^{\mathrm{n}}(X) \simeq \mathrm{Spec}\, R$,
(2) the $\mathbb{P}^1$-bundles $\tilde{U}_i = \mathbb{P}(\tilde{p}_* \mathcal{O}_{\tilde{U}_i}(\bar{\sigma}_i)^\vee)$, for $i = 1, 2$ are trivial, and
(3) there exists a Cartier divisor $\tau \subset U$ such that $\iota^{-1}(\tau) = \tau_1 \cup \tau_2$, where $\tau_i \subset \tilde{U}_i$ are sections that are disjoint from $\bar{\sigma}_i$.

We can then find homogeneous bundle coordinates $[x_0 : x_1]$ on $\tilde{U}_1$ and $[y_0 : y_1]$ on $\tilde{U}_2$ such that

$$\bar{\sigma}_1 = \{([x_0 : x_1], b) \in \tilde{U}_1 \mid x_0 = 0\}, \qquad \tau_1 = \{([x_0 : x_1], b) \in \tilde{U}_1 \mid x_1 = 0\},$$
$$\bar{\sigma}_2 = \{([y_0 : y_1], b) \in \tilde{U}_2 \mid y_0 = 0\}, \text{ and} \qquad \tau_2 = \{([y_0 : y_1], b) \in \tilde{U}_2 \mid y_1 = 0\}.$$

If we set

$$\tilde{U}_0 := \tilde{U} \setminus (\tau_1 \cup \tau_2),$$

then the image $U_0 := \iota(\tilde{U}_0)$ is affine, and we can write the relevant modules as

$$\mathcal{O}_{\tilde{U}}(\tilde{U}_0) \simeq R \otimes (k[x_0] \oplus k[y_0])$$
$$\mathcal{O}_{\tilde{\sigma}_1}(\tilde{U}_0) \simeq R \otimes k[x_0]/(x_0^2).$$
$$\mathcal{O}_{\tilde{\sigma}_2}(\tilde{U}_0) \simeq R \otimes k[y_0]/(y_0^2).$$

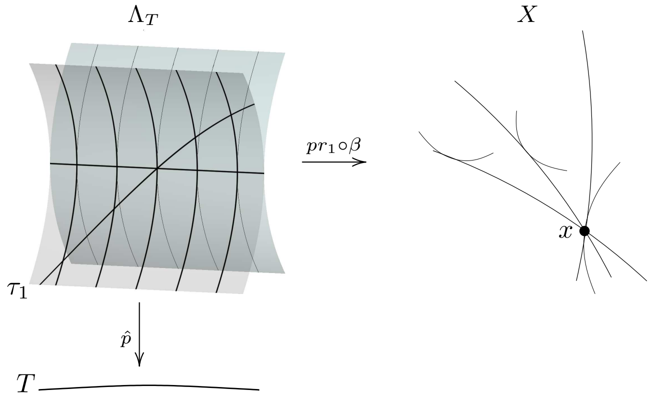


FIGURE 4.1. Proof of Theorem 1.4

Adjusting the bundle coordinates, if necessary, we can assume that the identification morphism $\gamma^{\#}(U_0) : \mathcal{O}_{\tilde{\sigma}_2}(\tilde{U}_0) \to \mathcal{O}_{\tilde{\sigma}_1}(\tilde{U}_0)$ is written as

$$\begin{array}{rccc} \gamma^{\#}(U_0): & R\otimes k[y_0]/(y_0^2) & \to & R\otimes k[x_0]/(x_0^2) \\ & r\otimes y_0 & \mapsto & r\otimes x_0. \end{array}$$

In this setup, we can find the morphism φ explicitly:

$$\begin{array}{rccc} \varphi(U_0): & R\otimes (k[x_0]\oplus k[y_0]) & \to & R\otimes k[x_0]/(x_0^2) \\ & r\otimes (f(x_0), g(y_0)) & \mapsto & r\otimes (f(x_0)-g(x_0)). \end{array}$$

Therefore, as an R-algebra, $\ker(\varphi)(U_0)$ is generated by the elements $u := 1_R \otimes (x_0, y_0)$ and $v := 1_R \otimes (x_0^2, 0)$, which satisfy the single relation $v(u^2 - v) = 0$. Thus

$$\ker(\varphi)(U_0) = R \otimes k[u, v]/(v(u^2 - v)).$$

In other words, $\beta^{-1}(U_0)$ is a bundle of two affine lines over $\mathrm{Dubbies}^{\mathrm{n}}(X)$, meeting tangentially in a single point.

It follows directly from the construction of $\mathcal{A}$ that α is an isomorphism away from $\bar{\sigma}_1 \cup \bar{\sigma}_2$. The curve $\hat{p}^{-1}(\ell)$ is therefore smooth outside of $\hat{p}^{-1}(\ell) \cap \beta^{-1}(U_0)$, and it follows that $\hat{p}^{-1}(\ell)$ is indeed a dubby. This shows that Λ is a bundle of dubbies.

To finish the proof, we need to verify that Λ_D is reducible. To that end, recall from section 3.3.2 that the universal family $\tilde{U}|_D = \tilde{p}^{-1}(D)$ is the disjoint union of two $\mathbb{P}_1$-bundles. Since α is isomorphic away from $\bar{\sigma}_1 \cup \bar{\sigma}_2$, it follows that $\Lambda_D = \alpha(\tilde{U}|_D)$ is reducible as claimed. This ends the proof. $\square$

4. PROOFS OF THE MAIN THEOREMS

4.1. **Proof of Theorem 1.3.** The assertion that all curves associated with H_x are smooth at a general point $x \in X$ follows immediately from the assumption that none of the curves $\ell \in H$ is cuspidal, and by [Keb02b, thms. 2.4(1) and 3.3(1)]. It remains to show that no two curves intersect tangentially.

We will argue by contradiction and assume that we can find a pair $\ell = \ell_1 \cup \ell_2 \subset X$ of distinct curves $\ell_i \in H$ that intersect tangentially at x. The pair ℓ is then dominated by a dubby whose singular point maps to x. Loosely speaking, we will move the point of intersection to obtain a positive-dimensional family of dubbies that all contain the point x –see figure 4.1.

Setup. To formulate our setup more precisely, we will use the notation introduced in diagram (3.1) of Proposition 3.6 and recall from Proposition 3.4 that $\mathrm{Dubbies}^{\mathrm{n}}(X)|_H$ is proper. Recall further that the universal family U is a subset $U \subset X \times \mathrm{Dubbies}^{\mathrm{n}}(X)$ and let $\jmath := pr_1 \circ \iota : \tilde{U} \to X$ be the canonical morphism. The assumption that for every general point $x \in X$, there is a pair of curves intersecting tangentially at x can be reformulated as

$$\jmath \circ \sigma_1(\mathrm{Dubbies}^{\mathrm{n}}(X)|_H) = \jmath \circ \sigma_2(\mathrm{Dubbies}^{\mathrm{n}}(X)|_H) = X$$

Let $D \subset \mathrm{Dubbies}^{\mathrm{n}}(X)|_H$ be an irreducible component such that

$$\jmath \circ \sigma_1(D) = \jmath \circ \sigma_2(D) = X$$

holds. By abuse of notation, we will denote $\tilde{U}_D = (\tilde{U}_D)_1 \cup (\tilde{U}_2)_D$ by $\tilde{U} = \tilde{U}_1 \cup \tilde{U}_2$. Fix a closed point $t \in D$ and consider the intersection numbers

$$d_1 := \jmath^*(L) \cdot (\tilde{p}^{-1}(t) \cap \tilde{U}_1) \quad \text{and} \quad d_2 := \jmath^*(L) \cdot (\tilde{p}^{-1}(t) \cap \tilde{U}_2)$$

Renumbering $\tilde{U}_1$ and $\tilde{U}_2$, if necessary, we may assume without loss of generality that $d_1 \geq d_2$. In this setup it follows from the upper semi-continuity of the fiber dimension that $(\jmath|_{\tilde{U}_1})^{-1}(x)$ contains an irreducible curve τ_1 which intersects $\sigma_1(D)$ non-trivially and is not contained in

$$S := \{y \in \tilde{U} \mid \iota \text{ is not an isomorphism at } y\}$$

Set $T := \tilde{p}(\tau_1)$. After a base change, if necessary, we may assume that T is a normal curve and consider the restrictions of the morphisms constructed in Proposition 3.6:

$$\begin{tikzcd} \tilde{U}_T \arrow[r, "\alpha"'] \arrow[rrr, bend left, "\jmath"] & \Lambda_T \arrow[r, "\beta"'] & U_T \arrow[r, "pr_1"'] & X \end{tikzcd}$$

Using [Keb02a, thm. 3.3.(1)], we find that τ_1 is generically injective over T, and therefore is a section. Let $\tilde{U}_{T,1} = (\tilde{U}_1)_T$ and $\tilde{U}_{T,2} = (\tilde{U}_2)_T$. It follows directly from the reducibility assertion of Proposition 3.6 Λ_T is reducible, and it follows from Proposition 3.3 that $(\tilde{U}_{T,1}, \sigma_1(T))$ and $(\tilde{U}_{T,2}, \sigma_2(T))$ are isomorphic pairs over T. Let $\gamma : \tilde{U}_{T,1} \to \tilde{U}_{T,2}$ be an isomorphism and consider the section $\tau_2 := \gamma(\tau_1) \subset \tilde{U}_{T,2}$.

The contraction of τ_2. With the notation above, Theorem 1.3 follows almost immediately from the following observation.

Lemma 4.1. *The morphism $\jmath$ contracts the section τ_2 to x, i.e., $\tau_2 \subset \jmath^{-1}(x)$.*

Notice that this finishes the proof of Theorem 1.3. Indeed, Lemma 4.1 implies that a general point $t \in T$ corresponds to a pair $\ell_t = \ell_{t,1} \cup \ell_{t,2}$ of two distinct curves that intersect at x. The curve ℓ_t is then singular at x, a contradiction to the fact that $\tau_1 \not\subset S$.

Proof Lemma 4.1. As a first step, we show that $\jmath$ contracts τ_2 to some point $y \in X$. The proof relies on a calculation of intersection numbers on the ruled surfaces $\tilde{U}_{T,1}$ and $\tilde{U}_{T,2}$. Recall the basic fact that

$$\mathrm{Num}(\tilde{U}_{T,1}) = \mathbb{Z} \cdot \sigma_1(T) \oplus \mathbb{Z} \cdot F_{V,1}$$

where $F_{V,1}$ is a fiber of $\tilde{p}_{\tilde{U}_{T,1}} : \tilde{U}_{T,1} \to T$. A similar decomposition holds for $\tilde{U}_{T,2}$. Since τ_1 is a section, we have the numerical equivalence,

$$\tau_1 \equiv \sigma_1(T) + d \cdot F_{V,1},$$

where d is a suitable integer. Since γ maps $\sigma_1(T)$ isomorphically onto $\sigma_2(T)$, we obtain a similar equation on $\tilde{U}_{T,2}$,

$$\tau_2 \equiv \sigma_2(T) + d \cdot F_{V,2}.$$

Next take the ample line bundle $L \in \mathrm{Pic}(X)$ and set

$$d_3 := \jmath^*(L) \cdot \sigma_1(T) = \jmath^*(L) \cdot \sigma_2(T).$$

These two numbers are indeed equal since the evaluation morphism identifies the images of the two sections $\sigma_1(T)$ and $\sigma_2(T)$. Now we can write the intersection numbers as

$$\begin{aligned} \jmath^*(L) \cdot \tau_2 &= \jmath^*(L) \cdot (\sigma_2(T) + d \cdot F_{V,2}) \\ &= d_3 + d \cdot d_2 = \underbrace{(d_3 + d \cdot d_1)}_{=\jmath^*(L)\cdot\tau_1 = 0} + d \cdot (d_2 - d_1) \\ &= d \cdot (d_2 - d_1) \leq 0 \end{aligned}$$

Since L is ample, this shows that $\jmath(\tau_2)$ is a point, $y \in X$.

It remains to prove that $x = y$. In order to see that, it suffices to recall two facts. First, as it was already used above, the evaluation morphism identifies the images of the two sections $\sigma_1(T)$ and $\sigma_2(T)$. Second, we know that τ_1 and the canonical section $\sigma_1 \subset \Lambda_1$ intersect. Let $t \in \tilde{p}(\tau_1 \cap \sigma_1(T))$ be a closed point. The two sections τ_2 and $\sigma_2(T)$ will then also intersect, $t \in \tilde{p}(\tau_2 \cap \sigma_2(T))$ and we obtain

$$\begin{aligned} x = \jmath(\tau_1) &= \jmath(\tau_1 \cap \sigma_1(T) \cap \tilde{p}^{-1}(t)) \\ &= \jmath(\tau_2 \cap \sigma_2(T) \cap \tilde{p}^{-1}(t)) = y. \end{aligned}$$

Lemma 4.1 is thus shown. □

4.2. **Proof of Theorem 1.4.** Let $H \subset \mathrm{RatCurves}^{\mathrm{n}}(X)$ be as in Theorem 1.4. We assume without loss of generality that all irreducible components of H dominate X. Fix an ample line bundle $L \in \mathrm{Pic}(X)$ and let $H' \subset H$ be an irreducible component such that for a general curve $\mathcal{C} \in H'$ the intersection number $L.\mathcal{C}$ is minimal among all the intersection numbers of L with curves from H. Finally, fix a rational curve $\mathcal{C} \subset X$ that corresponds to a general point of H'.

The proof of Theorem 1.4 now follows very much the lines of the proof of Theorem 1.3 from the previous section. The main difference to the previous argument is that we have to work harder to find the family T, as the properness of $\mathrm{Dubbies}^{\mathrm{n}}(X)|_H$ is no longer automatically guaranteed. Over the complex number field, however, the following lemma holds, which replaces the properness assumption in our context.

Lemma 4.2. *Assume that X is a complex-projective manifold, and let $S' \subset X$ be a subvariety of codimension $\mathrm{codim}_X S' \geq 2$. If $\mathcal{C} \in H$ is a curve that corresponds to a general point of H', then $\mathcal{C}$ and S' are disjoint: $\mathcal{C} \cap S' = \emptyset$.*

Proof. [Kol96, Chapt. II, Prop. 3.7 and Thm. 3.11] □

Corollary 4.3. *Under the assumptions of Theorem 1.4, if $\mathcal{C} \in H'$ is a general curve, and if $\mathrm{codim}_X D \geq 2$, then*

$$H_{\mathcal{C}} := \{\mathcal{C}' \in H \,|\, \mathcal{C} \cap \mathcal{C}' \neq \emptyset\} \subset H$$

is proper, and the associated curves are immersed along $\mathcal{C}$. In particular, $\mathcal{C}$ is immersed.

Proof. It suffices to note that $\mathcal{C}$ is disjoint from both S and D. □

Before coming to the proof of Theorem 1.4, we give a last preparatory lemma concerning the dimension of the locus D of cusps.

Lemma 4.4. *If $D \subset X$ is a divisor, then the subfamily $H^{cusp} \subset H$ of cuspidal curves dominates X.*

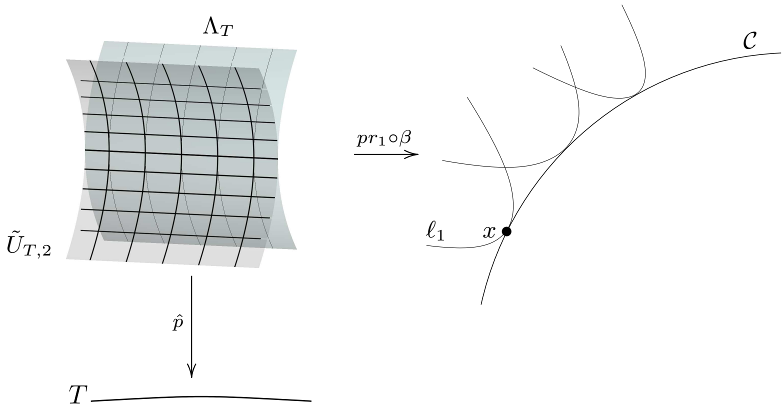


FIGURE 4.2. Proof of Theorem 1.4

Proof. Argue by contradiction and assume that all cuspidal curves in H^{cusp} are contained in a divisor. The total space of the family of cuspidal curves is then at least $(\dim D + 1)$-dimensional, so for a general point $x \in D$ there exists a positive dimensional family of cuspidal curves that contain x and are contained in D. That, however, is impossible: it has been shown in [Keb02a, Thm. 3.3] that in the projective variety D, a general point is contained in no more then finitely many cuspidal curves. $\square$

Setup of the proof. For the proof of Theorem 1.4, we will again argue by contradiction. By Lemma 4.4 this amounts to the assumption that τ is not generically injective, and that $\operatorname{codim}_X D \geq 2$. By Corollary 4.3, this implies that the space of curves which intersect $\mathcal{C}$ is proper and all associated curves are immersed along $\mathcal{C}$. Since $\mathcal{C}$ was a general curve, the assumptions also imply that for a general point $x \in \mathcal{C}$, there exists a point $t \in \mathrm{Dubbies}^{\mathrm{n}}(X)$ corresponding to a pair of marked curves $\ell = \ell_1 \cap \ell_2$ such that $\ell_2 = \mathcal{C}$ and ℓ_1 intersects $\mathcal{C}$ tangentially at x, i.e., $\mathrm{Image}(\tau(\sigma_1(t))) = \mathbb{P}(T_{\mathcal{C}}|_x^\vee)$ where $\tau : \tilde{U} \to \mathbb{P}(T_X^\vee)$ is the tangent morphism from the introduction. Hence we can find a proper curve $T \subset \mathrm{Dubbies}^{\mathrm{n}}(X)$ with associated diagram

$$\begin{tikzcd} \tilde{U}_T \arrow[r, "\alpha"'] \arrow[rrr, bend left, "\jmath"] \arrow[dr, "\tilde{p}"'] & \Lambda_T \arrow[r, "\beta"'] \arrow[d, "\hat{p}"] & U_T \arrow[r, "pr_1"'] \arrow[dl, "p"] & X \\ & T & & \end{tikzcd}$$

such that $\tilde{U}_T$ decomposes as $\tilde{U}_T = \tilde{U}_{T,1} \cup \tilde{U}_{T,2}$, where

$$\tilde{U}_{T,2} \simeq \tilde{\mathcal{C}} \times T \simeq \mathbb{P}^1 \times T,$$

and where $\tau|_{\sigma_1(T)}$ dominates $\mathbb{P}(T_{\mathcal{C}}^\vee)$.

End of proof. We are now in a situation which is very similar to the one considered in the proof of Theorem 1.3: we will derive a contradiction by calculating certain intersection numbers on $\tilde{U}_{T,1}$ and $\tilde{U}_{T,2}$.

As a first step, remark that $\tilde{U}_{T,1}$ maps to a surface in X. It follows that $\jmath^*(L)$ is nef and big on $\tilde{U}_{T,1}$.

Secondly, since $\tilde{U}_{T,2}$ is isomorphic to the trivial bundle $\mathbb{P}^1 \times T$, we have a decomposition

$$\mathrm{Num}(\tilde{U}_{T,2}) \simeq \mathbb{Z} \cdot F_{H,2} \oplus \mathbb{Z} \cdot F_{V,2}$$

where $F_{H,2}$ is the numerical class of a fiber of the map $\tilde{U}_{T,2} \to \mathbb{P}^1$ and $F_{V,2}$ that of a fiber of the map $\tilde{U}_{T,2} \to T$. Likewise, since the pairs $(\tilde{U}_{T,1}, \sigma_1(T))$ and $(\tilde{U}_{T,2}, \sigma_2(T))$ are isomorphic, let

$$\mathrm{Num}(\tilde{U}_{T,1}) \simeq \mathbb{Z} \cdot F_{H,1} \oplus \mathbb{Z} \cdot F_{V,1}$$

be the corresponding decomposition. If d denotes the degree of the (finite, surjective) morphism

$$\jmath \circ \sigma_1 = \jmath \circ \sigma_2 : T \to \mathcal{C},$$

then it follows directly from the construction that the curves of type $F_{H,2}$ intersect $\sigma_2(T)$ with multiplicity d. We obtain that

$$\sigma_2(T) \equiv F_{H,2} + d \cdot F_{V,2} \quad \text{and thus} \quad \sigma_1(T) \equiv F_{H,1} + d \cdot F_{V,1}.$$

To end the argumentation, let

$$d_1 := \jmath^*(L) \cdot F_{V,1} \quad \text{and} \quad d_2 := \jmath^*(L) \cdot F_{V,2}$$

In particular, since $\jmath^*(L).F_{H,2} = 0$, we have that $\jmath^*(L) \cdot \sigma_2 = d \cdot d_2$. Recall that $H' \subset H$ was chosen so that $d_1 \geq d_2$ and write:

$$\begin{aligned} \jmath^*(L) \cdot F_{H,1} &= \jmath^*(L) \cdot (\sigma_1(T) - d \cdot F_{V,1}) \\ &= d \cdot d_2 - d \cdot d_1 \\ &\leq 0. \end{aligned}$$

Because $\tilde{U}_{T,1}$ is covered by curves which are numerically equivalent to $F_{H,1}$ that contradicts the assumption that $\jmath^*(L)|_{\tilde{U}_{T,1}}$ is big and nef. The proof of Theorem 1.4 is thus finished. □

5. Applications

5.1. **Irreducibility Questions.** Let $H \subset \mathrm{RatCurves}^{\mathrm{n}}(X)$ be a maximal dominating family of rational curves of minimal degrees on a projective variety X. How many components can H have? If we pick an irreducible component $H' \subset H$ and fix a general point $x \in X$, does it follow that

$$H'_x := \{\ell \in H' \,|\, x \in \ell\}$$

is irreducible? These questions have haunted the field for quite a while now, as the possibility that H'_x might be reducible poses major problems in many of the proposed applications of rational curves to complex geometry —see the discussion in [Hwa01].

It is conjectured [Hwa01, chap. 5, question 2] that the answers to both of these questions are affirmative for a large class of varieties. There exists particularly strong evidence if X is a complex manifold and if the dimension of H'_x is not too small. Theorem 1.3 enables us to give a partial answer.

Theorem 5.1. *Under the assumptions of Theorem 1.3, if X is a complex manifold and if for a general point $x \in X$, and for all irreducible components $H' \subset H$*

$$\dim H'_x \geq \frac{\dim X - 1}{2},$$

then H_x is irreducible. In particular, H is irreducible.

The main technical difficulty in proving Theorem 5.1 lies in the fact that the closed points of H are generally not in 1:1-correspondence with actual rational curves, a possibility that is sometimes overlooked in the literature. As a matter of fact, this correspondence is only generically injective, and it may well happen that two or more points of H correspond to the same curve $\ell \subset X$. This is due to the very construction of the space $\mathrm{RatCurves}^{\mathrm{n}}(X)$: recall from section 3.3 that $\mathrm{RatCurves}^{\mathrm{n}}(X)$ is constructed as the quotient of the *normalization* of $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$. While $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$ is in 1:1-correspondence with morphisms, $\mathbb{P}^1 \to X$, that are birational onto their imnage, the normalization morphism

$$\mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X) \to \mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X),$$

need not be injective. For complex manifolds, however, we have the following workaround.

Lemma 5.2. *Under the assumptions of Theorem 5.1, let $x \in X$ be a general point and set*

$$H_x := \{\ell \in H \mid x \in \ell\}$$

Then the closed points of H_x are in 1:1-correspondence with the associated curves in X.

Proof. Since x is a general point and since we have picked a fixed component, H', all rational curves through x are free by the proof of [KMM92, thm. 1.1]. The space $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$ is therefore smooth at every point $f \in \mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$ whose image contains the point x by [Kol96, thm. II.1.7]. The normalization morphism

$$\mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X) \to \mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X),$$

is thus isomorphic in a neighborhood of f. Since $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$ is in 1:1 correspondence with morphisms $\mathbb{P}^1 \to X$, the claim follows. $\square$

This enables us to prove Theorem 5.1.

Proof of Theorem 5.1. Choose a general point $x \in X$, and let $\tau : H \dashrightarrow \mathbb{P}(T_X^\vee)$ be the tangent morphism described in the introduction. Since all curves associated with H_x are smooth, τ restricts to a regular morphism

$$\tau_x : H_x \to \mathbb{P}(T_X^\vee|_x).$$

This morphism is known to be finite [Keb02a, thm. 3.4]. By Theorem 1.3, τ_x is injective.

Now assume that H_x is not irreducible, $H_x = H_{x,1} \cup \ldots \cup H_{x,n}$. Since τ_x is finite, we have that

$$\dim(\tau_x(H_{x,1})) + \dim(\tau_x(H_{x,2})) \geq \dim X - 1 = \dim \mathbb{P}(T_X^\vee|_x)$$

Thus, by [Har77, thm. I.7.2],

$$\tau_x(H_{x,1}) \cap \tau_x(H_{x,2}) \neq \emptyset.$$

It follows that τ is not injective, a contradiction. $\square$

Lemma 5.2 raises the following question.

Question 5.3. *Are there other conditions than smoothness over $\mathbb{C}$ which guarantee that closed points of H_x are in 1:1-correspondence with rational curves?*

5.2. **Automorphism groups of projective manifolds and their spaces of rational curves.** The setup of Theorem 5.1 naturally generalizes the notion of a prime Fano manifold, i.e., one that is covered by lines under a suitable embedding. Some of the results that have been obtained for prime Fanos hold in the more general setup of Theorem 5.1. We give one example.

For any complex variety X, let $\mathrm{Aut}_0(X)$ denote the maximal connected subgroup of the group of automorphisms. By universal properties, an automorphism of a complex variety induces an automorphism of the space $\mathrm{RatCurves}^{\mathrm{n}}(X)$. It might be interesting to note that in our setup the converse also holds.

Theorem 5.4. *In the setup of Theorem 5.1, if* $b_2(X) = 1$*, then the groups* $\mathrm{Aut}_0(X)$ *and* $\mathrm{Aut}_0(H)$ *coincide.*

Proof. The theorem follows from Theorem 5.1 and [HM02, Thm. 1] —observe that the proof of [HM02, Thm. 1] works without the assumption that H is a dominating family of rational curves of minimal degrees because we assume here that H is proper. □

5.3. **Contact Manifolds.** Let X be a projective contact manifold over $\mathbb{C}$, e.g. the twistor space over a Riemannian manifold with Quaternionic-Kählerian holonomy group and positive curvature. We refer to [Keb01c] and the references therein for an introduction and for the relevant background information.

If X is different from the projective space, it has been shown in [Keb01c] that X is covered by a compact family of rational curves $H \subset \mathrm{RatCurves}^{\mathrm{n}}(X)$ such that for a general point x, all curves associated with points in H_x are smooth. Thus, the assumptions of Theorem 1.4 are satisfied, and τ is generically injective. This has been shown previously in [Keb01b] using rather involved arguments which heavily rely on obstructions to deformations coming from contact geometry.

REFERENCES

[BBI00] L. Bădescu, M.C. Beltrametti and P. Ionescu. Almost-lines and quasi-lines on projective manifolds. In T. Peternell and F.O. Schreyer, editors, *Complex Analysis and Algebraic Geometry*, pages 1–27. de Gruyter, 2000.

[CMSB00] K. Cho, Y. Miyaoka, and N.I. Shepherd-Barron. Characterizations of Projective Spaces and Applications. Preprint, October–December 2000.

[Eis95] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer, 1995.

[Har77] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, 1977.

[Hwa01] J.-M. Hwang. Geometry of Minimial Rational Curves on Fano Manifolds ICTP Lecture Notes Series Volume VI (ISBN 92-95003-09-8), December 2001. Available on the internet at `http://www.ictp.trieste.it/~pub_off/lectures/vol6.html`

[HM02] J.-M. Hwang and N. Mok. Automorphism groups of the spaces of minimal rational curves on Fano manifolds of Picard number 1, Preprint, to appear.

[Keb01a] S. Kebekus. Rationale Kurven auf projektiven Mannigfaltigkeiten (German). Habilitationsschrift, February 2001. Available from the author's home page at `http://btm8x5.mat.uni-bayreuth.de/~kebekus`

[Keb01b] S. Kebekus. Lines on Contact Manifolds II. LANL-Preprint math.AG/0103208, 2001.

[Keb01c] S. Kebekus. Lines on contact manifolds. *J. Reine Angew. Math*, 539:167–177, October 2001.

[Keb02a] S. Kebekus. Families of singular rational curves. *J. Alg. Geom.*, 11:245–256, 2002.

[Keb02b] S. Kebekus. Characterizing the projective space after Cho, Miyaoka and Shepherd-Barron. In I. Bauer, F. Catanese, Y. Kawamata, T. Peternell, and Y.-T. Siu, editors, *Complex Geometry, Collection of Papers dedicated to Hans Grauert*, pages 147–156. Springer, 2002.

[KMM92] J. Kollár, Y. Miyaoka, and S. Mori. Rational Connectedness and Boundedness of Fano Manifolds. *J. Diff. Geom.*, 36:765–769, 1992.

[Kol96] J. Kollár. *Rational Curves on Algebraic Varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge*. Springer, 2nd edition, 1996.

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