Preface

This book is an introduction to manifolds at the beginning graduate level. It contains the essential topological ideas that are needed for the further study of manifolds, particularly in the context of differential geometry, algebraic topology, and related fields. Its guiding philosophy is to develop these ideas rigorously but economically, with minimal prerequisites and plenty of geometric intuition. Here at the University of Washington, for example, this text is used for the first third of a year-long course on the geometry and topology of manifolds; the remaining two-thirds focuses on smooth manifolds.

There are many superb texts on general and algebraic topology available. Why add another one to the catalog? The answer lies in my particular vision of graduate education—it is my (admittedly biased) belief that every serious student of mathematics needs to know manifolds intimately, in the same way that most students come to know the integers, the real numbers, Euclidean spaces, groups, rings, and fields. Manifolds play a role in nearly every major branch of mathematics (as I illustrate in Chapter 1), and specialists in many fields find themselves using concepts and terminology from topology and manifold theory on a daily basis. Manifolds are thus part of the basic vocabulary of mathematics, and need to be part of the basic graduate education. The first steps must be topological, and are embodied in this book; in most cases, they should be complemented by material on smooth manifolds, vector fields, differential forms, and the like. (After all, few of the really interesting applications of manifold theory are possible without using tools from calculus.)

Of course, it is not realistic to expect all graduate students to take full-year courses in general topology, algebraic topology, and differential geometry. Thus, although this book touches on a generous portion of the material that is typically included in much longer courses, the coverage is selective and relatively concise, so that most of the book can be covered in a single quarter or semester, leaving time in a year-long course for further study in whatever direction best suits the instructor and the students. At U.W. we follow it with a two-quarter sequence on smooth manifold theory; but it could equally well lead into a full-blown course on algebraic topology.

It is easy to describe what this book is not. It is not a course on general topology—many of the topics that are standard in such a course are ignored here, such as metrization theorems; infinite products and the Tychonoff theorem; countability and separation axioms and the relationships among them (other than second countability and the Hausdorff axiom, which are part of the definition of manifolds); and function spaces. Nor is it a course in algebraic topology—although I treat the fundamental group in detail, there is barely a mention of the higher homotopy groups, and the treatment of homology theory is extremely brief, meant mainly to give the flavor of the theory and to lay some groundwork for the later introduction of de Rham cohomology. It is certainly not a comprehensive course on topological manifolds, which would have to include such topics as PL structures and maps, transversality, intersection theory, cobordism, bundles, characteristic classes, and low-dimensional geometric topology. Finally, it is not intended as a reference book, because few of the results are presented in their most general or most complete form.

Perhaps the best way to summarize what this book is would be to say that it represents, to a good approximation, my conception of the ideal amount of topological knowledge that should be possessed by beginning graduate students who are planning to go on to study smooth manifolds and differential geometry. Experienced mathematicians will probably observe that my choices of material and approach have been influenced by the fact that I am a differential geometer and analyst by training and predilection, not a topologist. Thus I give special emphasis to topics that will be of importance later in the study of smooth manifolds, such as group actions, orientations, and degree theory. (A few topological ideas that are important for manifold theory, such as paracompactness and embedding theorems, are omitted because they are better treated in the context of smooth manifolds.) But despite my prejudices, I have tried to make the book useful as a precursor to algebraic topology courses as well, and it could easily serve as a prerequisite to a more extensive course in homology and homotopy theory.

Prerequisites. The prerequisite for studying this book is, briefly stated, a solid undergraduate degree in mathematics; but this probably deserves some elaboration. Traditionally, “algebraic topology” has been seen as a separate subject from "general topology," and most courses in the former begin with the assumption that the students have already completed a course in the latter. However, the sad fact is that for a variety of reasons, many undergraduate mathematics majors in the U.S. never take a course in general topology. For that reason I have written this book without assuming that the reader has had any exposure to topological spaces. On the other hand, I do assume several essential prerequisites beyond calculus and linear algebra: basic logic and set theory such as what one would encounter in any rigorous undergraduate analysis or algebra course; real analysis at the level of Rudin's *Principles of Mathematical Analysis* [Rud76], including, in particular, a thorough understanding of metric spaces and their continuous functions and compact subsets; and group theory at the level of Hungerford's *Abstract Algebra: An Introduction* [Hun90] or Herstein's *Topics in Algebra* [Her75]. Because it is vitally important that the reader be comfortable with this prerequisite material, I have collected in the Appendix a summary of the main points that are used throughout the book, together with a representative collection of exercises. These exercises, which should be relatively straightforward for anyone who has had the prerequisite courses, can be used by the student to refresh his or her knowledge, or can be assigned by the instructor at the beginning of the course to make sure that everyone starts with the same background.

Organization. The book is divided into thirteen chapters, which can be grouped into an introduction and five major substantive sections.

The introduction (Chapter 1) is meant to whet the student's appetite and create a "big picture" into which the many details can later fit.

The first major section, Chapters 2 through 4, is a brief and highly selective introduction to the ideas of general topology: topological spaces; their subspaces, products, and quotients; and connectedness and compactness. Of course, manifolds are the main examples and are emphasized throughout. These chapters emphasize the ways in which topological spaces differ from the more familiar Euclidean and metric spaces, and carefully develop the machinery that will be needed later, such as quotient maps, local path connectedness, and locally compact Hausdorff spaces.

The second major section, comprising Chapters 5 and 6, explores in detail the main examples that motivate the rest of the theory: simplicial complexes, 1-manifolds, and 2-manifolds. Chapter 5 introduces simplicial complexes in two ways—first concretely, as locally finite collections of simplices in Euclidean space that intersect nicely; and then abstractly, as collections of finite vertex sets. Both approaches are useful: The concrete definition helps students develop their geometric intuition, while the abstract point of view emphasizes the fact that all statements about simplicial complexes can be reduced to combinatorics. There are several reasons for introducing simplicial complexes at this stage: They furnish a rich source of examples; they give a very concrete way of thinking about orientations and the Euler characteristic; they provide the concept of triangulability needed for the classifications of 1-manifolds and 2-manifolds; and they set the stage for the treatment of homology later. Chapter 6 begins by proving a classification theorem for 1-manifolds using the triangulability theorem proved in the preceding chapter. The rest of the chapter is devoted to a detailed study of 2-manifolds. After exploring the basic examples of surfaces—the sphere, the torus, the projective plane, and their connected sums—I give a complete proof of the classification theorem for compact surfaces, essentially following the treatment in [Mas89].

The third major section, Chapters 7 through 10, is the core of the book. In it, I give a fairly complete and traditional treatment of the fundamental group. Chapter 7 introduces the definitions and proves the topological and homotopy invariance of the fundamental group. At the end of the chapter I insert a brief introduction to category theory. Categories are not used in a central way anywhere in the book, but it is natural to introduce them after having proved the topological invariance of the fundamental group, and it is useful for students to begin thinking in categorical terms early. Chapter 8 gives a detailed proof that the fundamental group of the circle is infinite cyclic. Because the techniques used here are the precursor and motivation for the entire theory of covering spaces, I introduce some of the terminology of the latter subject—evenly covered neighborhoods, local sections, lifting—in the special case of the circle, and the proofs here form a model for the proofs of more general theorems involving covering spaces to come in a later chapter. Chapter 9 is a brief digression into group theory. Although a basic acquaintance with group theory is an essential prerequisite, most undergraduate algebra courses do not treat free products, free groups, presentations of groups, or free abelian groups, so I develop these subjects from scratch. (The material on free abelian groups is included primarily for use in the treatment of homology in Chapter 13, but some of the results play a role also in classifying the coverings of the torus in Chapter 12.) The last chapter of this section gives the statement and proof of the Seifert–Van Kampen theorem, which expresses the fundamental group of a space in terms of the fundamental groups of its subsets, and describes several applications of the theorem including computation of the fundamental groups of graphs and of all the compact surfaces.

The fourth major section consists of two chapters on covering spaces. Chapter 11 defines covering spaces, gives a few examples, and develops the theory of the covering group. Much of the development goes rapidly here, because it is parallel to what was done earlier in the concrete case of the circle. The ostensible goal of Chapter 12 is to prove the classification theorem for coverings—that there is a one-to-one correspondence between isomorphism classes of coverings of X and conjugacy classes of subgroups of the fundamental group of X—but along the way two other ideas are developed that are of central importance in their own right. The first is the notion of the universal covering space, together with proofs that every manifold has a universal covering and that the universal covering space covers every other covering space. The second is the fact that the quotient of a manifold by a free, proper action of a discrete group yields a manifold. These ideas are applied to a number of important examples, including classifying coverings of the torus and the lens spaces, and proving that surfaces of higher genus are covered by the hyperbolic disk.

The fifth major section of the book consists of one chapter only, Chapter 13, on homology theory. In order to cover some of the most important applications of homology to manifolds in a reasonable time, I have chosen a "low-tech" approach to the subject. I focus mainly on singular homology because it is the most straightforward generalization of the fundamental group. After defining the homology groups, I prove a few essential properties, including homotopy invariance and the Mayer–Vietoris theorem, with a minimum of homological machinery. I could not resist including a (terribly brief) introduction to simplicial homology, just because it immediately yields the topological invariance of the Euler characteristic. The last section of the chapter is a brief introduction to cohomology, mainly with field coefficients, to serve as background for a treatment of de Rham theory in a later course. In keeping with the overall philosophy of focusing only on what is necessary for a basic understanding of manifolds, I do not even mention relative homology, homology with arbitrary coefficients, simplicial approximation, or the axioms for a homology theory.

Although this book grew out of notes designed for a one-quarter graduate course, there is clearly too much material here to cover adequately in ten weeks. It should be possible to cover all or most of it in a semester with well prepared students. The book could even be used for a full-year course, allowing the instructor to adopt a much more leisurely pace and to work out some of the problems as examples in class.

Each instructor will have his or her own ideas about what to leave out in order to fit the material into a short course. At the University of Washington, we typically do not cover the chapter on homology at all, and give short shrift to some of the simplicial theory and some of the more involved examples of covering maps. Others may wish to leave out some or all of the material on covering spaces, or the classification of surfaces. With students who have had a solid topology course, the first four chapters could be skipped or assigned as outside reading.

Exercises and Problems. As is the case with any new mathematical material, and perhaps even more than usual with material like this that is so different from the mathematics most students have seen as undergraduates, it is impossible to learn the subject without getting one's hands dirty and working out a large number of examples and problems. I have tried to give the reader ample opportunity to do so throughout the book. In every chapter, and especially in the early ones, there are "exercises" woven into the text. Do not ignore them; without their solutions, the text is incomplete.

The reader should take each exercise as a signal to stop reading, pull out a pencil and paper, and work out the answer before proceeding further. The exercises are usually relatively easy, and typically involve proving minor results or working out examples that are essential to the flow of the exposition. Some require techniques that the student probably already knows from prior courses; others ask the student to practice techniques or apply results that have recently been introduced in the text. A few are straightforward but rather long arguments that are more enlightening to work through on one's own than to read. In the later chapters, fewer things are singled out as exercises, but there are still plenty of omitted details in the text that the student should work out before going on; it is my hope that by the time the student reaches the last few chapters he or she will have developed the habit of stopping and working through most of the details that are not spelled out without having to be told.

At the end of each chapter is a selection of "problems." These are, with a few exceptions, harder and/or longer than the exercises, and give the student a chance to grapple with more significant issues. The results of a number of the problems are used later in the text. There are more problems than most students could do in a quarter or a semester, so the instructor will want to decide which ones are most germane and assign those as homework.

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Thanks are due also to Mary Sheetz, who did an excellent job producing some of the illustrations under the pressures of time and a finicky author.

My debt to the authors of several other textbooks will be obvious to anyone who knows those books: William Massey's *Algebraic Topology: An Introduction* [Mas89], Allan Sieradski's *An Introduction to Topology and Homotopy* [Sie92], Glen Bredon's *Topology and Geometry*, and James Munkres's *Topology: A First Course* [Mun75] and *Elements of Algebraic Topology* [Mun84] are foremost among them.

Finally, I would like to thank my wife, Pm, for her forbearance and unflagging support while I was spending far too much time with this book

and far too little with the family; without her help I unquestionably could not have done it.

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