

CORRECTIONS TO  
Riemannian Manifolds: An Introduction to Curvature

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*Changes or additions made in the past twelve months are dated.*

- **Page 15, Exercise 2.3:** The first sentence should read: “Suppose  $M \subset \widetilde{M}$  is a closed embedded submanifold.”

(6/2/09) **Page 16, second paragraph, Exercise 2.3(c):** Change  $\widetilde{X}f = 0$  to  $(\widetilde{X}f)|_M = 0$ .

- **Page 19, paragraph before Lemma 2.3:** Insert the following before the last sentence of the paragraph: “A *local frame* for  $E$  is a finite sequence  $(\sigma_1, \dots, \sigma_k)$  of smooth sections of  $E$  over  $U$  such that  $(\sigma_1|_p, \dots, \sigma_n|_p)$  form a basis for  $E_p$  at each point  $p \in U$ .”

- **Page 19, Lemma 2.3:**  $F_{i_1 \dots i_k}^{j_1 \dots j_l}$  should read  $F^i$ ; also, change the name of the local frame from  $\{E_i\}$  to  $\{\sigma_i\}$ .

- **Page 19, Exercise 2.4:** Replace the given exercise by:

- (a) If  $(\sigma_1, \dots, \sigma_k)$  is a local frame for a vector bundle  $E$  over an open set  $U \subset M$ , let  $\psi: U \times \mathbf{R}^k \rightarrow \pi^{-1}(U)$  be the map  $\psi(p, x) = x^i \sigma_i|_p$ . Show that  $\psi^{-1}$  is a local trivialization of  $E$ .
- (b) Prove Lemma 2.3.

- **Page 20, paragraph before Exercise 2.6:** Replace the first sentence by “Let  $(E_1, \dots, E_n)$  be any local frame for  $TM$ .”

- **Page 21, line 4:** Change  $F(X_1, \dots, X_k, \omega^1, \dots, \omega^l)$  to  $F(\omega^1, \dots, \omega^l, X_1, \dots, X_k)$ .

- **Page 21, just after Exercise 2.7:** Add the following sentence in a paragraph by itself: “Because of the result of Lemma 2.4, it is common to use the same symbol for both a tensor field and the multilinear map on sections that it defines, and to refer either of these objects as a tensor field.”

- **Page 24, third paragraph:** Change the last sentence to “(When  $M$  is connected, it can be shown that the isometry group is always . . . .)”

(6/7/09) **Page 25, second full paragraph, third line:** After “open subset of  $M$ ,” insert “and which is smooth as a map into  $M$ .”

- **Page 27, paragraph before Exercise 3.6:** Replace this paragraph by “The following exercise shows that the converse is true provided we make the additional assumption that  $\pi$  is a *normal* covering, which means that the group of covering transformations acts transitively on each fiber of  $\pi$ .”

- **Page 27, Exercise 3.6:** Change “smooth covering map” to “smooth normal covering map.”

- **Page 41, Exercise 3.11(iii):** Replace the last sentence by “In the higher-dimensional case, for any point  $p \in \mathbf{B}_R^n$  and any vector  $V \in T_p \mathbf{B}_R^n$ , first show that  $h_R^3(\kappa_* V, \kappa_* V) = h_R^2(V, V)$  if  $p \in \mathbf{B}_R^2 \subset \mathbf{B}_R^n$  and  $V$  is tangent to  $\mathbf{B}_R^2$ ; then show that the same is true if  $p \in \mathbf{B}_R^2$  but  $V$  is arbitrary (using the fact that  $h_R^3$  and  $h_R^2$  are multiples of the Euclidean metrics at  $p$  and  $\kappa(p)$ ); and finally conjugate  $\kappa$  with a suitable orthogonal transformation in  $n - 1$  variables to reduce to the case  $p \in \mathbf{B}_R^2$ .”

- **Page 46, Problem 3-9(a):** Change the problem statement to: “Note that the natural action of  $U(n + 1)$  on  $\mathbf{C}^{n+1}$  descends to a transitive action on  $\mathbf{CP}^n$ . Show that  $\mathbf{CP}^n$  can be uniquely given the structure of a smooth, compact, real  $2n$ -dimensional manifold on which this action is smooth.”

- **Page 58, Exercise 4.8:** At the end of the sentence, add “and the constant curves.”

(8/25/09) **Page 63, Problem 4-1:** Replace the last sentence of the problem by the following: “Show that the two sets of Christoffel symbols are related by the following formula:

$$\tilde{\Gamma}_{ij}^k = (A^{-1})_q^k (A_i^p E_p A_j^q + A_i^p A_j^r \Gamma_{pr}^q) .”$$

- **Page 63, problem 4-3(b):** Replace the first sentence by “Show that there are vector fields  $V$  and  $W$  on  $\mathbf{R}^2$  such that  $V = W = \partial_1$  along the  $x^1$ -axis, but the Lie derivatives  $\mathcal{L}_V(\partial_2)$  and  $\mathcal{L}_W(\partial_2)$  are not equal on the  $x^1$ -axis.”
- **Page 66, first full paragraph:** Second sentence should read “Any vector field on  $M$  can be extended to a smooth vector field on a neighborhood of  $M$  in  $\mathbf{R}^n$  by the result of Exercise 2.3(b).” The part of the second sentence after the two displayed equations should read “where  $X$  and  $Y$  are extended arbitrarily to a neighborhood of  $M$ , . . . .”
- **Page 66, proof of Lemma 5.1, second paragraph:** Second sentence should read “Let  $f \in C^\infty(M)$  be extended arbitrarily to a neighborhood of  $M$ .”
- **Page 76, commutative diagram:** Change  $T_p M$  and  $T_{\varphi(p)} \tilde{M}$  to  $\mathcal{E}_p$  and  $\mathcal{E}_{\varphi(p)}$ , respectively (the domains of the restricted exponential maps).

(8/25/09) **Page 78, Proposition 5.11(a):** Change “as long as  $\gamma_V$  stays within  $\mathcal{U}$ ” to “as long as  $t$  is in some interval  $J$  containing 0 such that  $\gamma_V(J) \subset \mathcal{U}$ .”

(3/26/09) **Page 78, line 6 from the bottom:** Change “thought” to “though.”

- **Page 86, last sentence:** Replace the first part of the sentence by “In the higher-dimensional case, we just precede  $\kappa$  with a suitable orthogonal transformation of the ball, and follow it with a translation and rotation in the  $x$  variables (both of which preserve geodesics as well as lines and circles), and apply the usual . . . .”
- **Page 88, Problem 5-6(b):** In the first displayed equation, replace  $\omega \otimes N$  by  $\omega \otimes N^b$ .
- **Page 89, Problem 5-9:** Insert the following sentence after line 3: “(If  $Z$  is any vector field on  $M$ , we are using the notation  $\tilde{Z}$  to denote its horizontal lift.)” Also, in the hint, change both occurrences of  $\nabla$  to  $\tilde{\nabla}$ .
- **Page 93, Exercise 6-3(a):** After “independent of parametrization,” add “in the following sense: If  $\varphi: [c, d] \rightarrow [a, b]$  is a smooth map with smooth inverse, then

$$\int_a^b f(t) |\dot{\gamma}(t)| dt = \int_c^d \tilde{f}(u) |\dot{\tilde{\gamma}}(u)| du,$$

where  $\tilde{f} = f \circ \varphi$  and  $\tilde{\gamma} = \gamma \circ \varphi$ .”

- **Page 95, second displayed inequality:** Delete “ $d(p, q) \geq$ ” from the beginning of the inequality, and replace the next sentence by “It follows that  $d(p, q) \geq c\varepsilon > 0$ , so  $d$  is a metric.”

(3/26/09) **Page 101, proof of Corollary 6.7, third line:** Change “second variation formula” to “first variation formula.”

(9/9/09) **Page 102, proof of Theorem 6.8, fifth line:** Change “ $R = d(p, q)$ ” to “ $R = |V|_g$ .”

- **Page 105, first sentence of last paragraph:** Change both instances of “[ $a, b$ ]” to “[ $0, b$ ].”

- **Page 111, Corollary 6.15:** Change the statement to “If  $M$  is complete, then any two points in  $M$  can be joined by a minimizing geodesic segment.”
- **Page 112, Problem 6-2:** Replace the hint by “[Hint: For the hard direction, proceed as follows. (1) Show that any metric isometry  $\varphi: (M, g) \rightarrow (\widetilde{M}, \widetilde{g})$  takes geodesics to geodesics. (2) For any  $p \in M$ , show that there is an open ball  $\mathcal{V} = B_\varepsilon(0) \subset T_p M$  and a map  $\psi: \mathcal{V} \rightarrow T_{\varphi(p)} \widetilde{M}$  satisfying  $\exp_{\varphi(p)} \psi(X) = \varphi(\exp_p X)$  for all  $X \in \mathcal{V}$ . (3) If  $\varepsilon$  is small enough and  $X, Y \in \mathcal{V}$ , show that there exists a constant  $C > 0$  such that

$$(1 - C|t|)|tX - tY|_g \leq d_g(\exp_p tX, \exp_p tY) \leq (1 + C|t|)|tX - tY|_g$$

whenever  $|t| \leq 1$ , by comparing  $g$  with the Euclidean metric in normal coordinates. (4) Using the result of (3), conclude that

$$\lim_{t \rightarrow 0} \frac{d_g(\exp_p tX, \exp_p tY)^2}{t^2} = |X - Y|_g^2 = |X|_g^2 + |Y|_g^2 - 2\langle X, Y \rangle_g,$$

and an analogous formula holds for  $\widetilde{g}$ . (5) Show that  $\langle \psi(X), \psi(Y) \rangle_{\widetilde{g}} = \langle X, Y \rangle_g$  for all  $X, Y \in \mathcal{V}$ . (6) Show that  $\psi$  is the restriction of a linear map. (7) Conclude that  $\varphi$  is smooth and  $\varphi_* = \psi$ .]”

- **Page 112, Problem 6-4:** In part (a), change the second sentence to “For  $\varepsilon > 0$  small enough that  $B_{3\varepsilon}(p) \subset \mathcal{W}, \dots$ ” In part (b), add to the hint “Be careful to verify that  $\varepsilon$  can be chosen independently of  $V$ .”
- **Page 113, Problem 6-8:** Delete the word “complete” and add instead “connected.” Also, revise the hint as follows: “[Hint: Given  $p, q \in M$  sufficiently near each other, consider the midpoint of a geodesic joining  $p$  and  $q$ . You may use without proof the fact that the isometry group of  $M$  is a Lie group acting smoothly on  $M$ .]”

(3/26/09) **Page 120, line above (7.5):** Change  $x^m$  to  $x^n$ .

- **Page 125, line 4:** Replace the phrase “where  $\operatorname{div}$  is the divergence operator (Problem 3-3)” by “where  $\operatorname{div} Rc$  is the 1-tensor obtained from  $\nabla Rc$  by raising one index and contracting.”

(7/2/09) **Page 137, last displayed equation:** Should be

$$\kappa = \left( \frac{|D_t \dot{\gamma}(t)|^2}{|\dot{\gamma}(t)|^4} - \frac{\langle D_t \dot{\gamma}(t), \dot{\gamma}(t) \rangle^2}{|\dot{\gamma}(t)|^6} \right)^{1/2}.$$

- **Page 139, line before Exercise 8.4:** Change “lies entirely in  $M$ ” to “lies in  $M$  at least for some small time interval  $(-\varepsilon, \varepsilon)$ .”
- **Page 150, Problem 8-3:** The left-hand side of the displayed equation should be  $h(V, V)$  instead of  $h(V, W)$ , and the denominator on the right-hand side should be  $|\operatorname{grad} F|$  instead of  $|\operatorname{grad} F|^2$ . Also, replace the last sentence with the following: “Show that the mean curvature of  $M$  is given by

$$H = -\frac{1}{n} \operatorname{div} \left( \frac{\operatorname{grad} F}{|\operatorname{grad} F|} \right) = -\frac{1}{n} \sum_{i,j=1}^{n+1} \frac{(\partial_i \partial_i F)(\partial_j F)(\partial_j F) - (\partial_i \partial_j F)(\partial_i F)(\partial_j F)}{|\operatorname{grad} F|^3}.$$

[Hint: Use an adapted orthonormal frame.]”

- **Page 151, Problem 8-6:** In the second to last line, change  $K dV_g$  to  $(-1)^n K dV_g$ .

(3/26/09) **Page 165, fourth display:** In some editions, there is a dot over  $\omega$  in the last integral in the first line of that display. It should not be there.

- **Page 171, Problem 9-3(b):** Change “equal interior angles” to “equal interior angles and proportional corresponding side lengths.” [The claim is true without this extra hypothesis, but the proof requires a more detailed analysis of hyperbolic and (especially) spherical geometry than is worth carrying out just for this problem.]
- **Page 176, Exercise 10.1:** This is somewhat harder than most of the other exercises in the book, and needs Proposition 10.4 for its solution, so it should probably be moved to the Problems section, say as Problem 10-4.
- **Page 180, statement of Proposition 10.9:** The first case of formula (10.8) should be  $C = 0$ , not  $K = 0$ .

(3/26/09) **Page 184, last line of the proof of Proposition 10.11:** Change  $J_W(q)$  to  $J_W(1)$ .

- **Page 188, Figure 10.10:** Replace  $\gamma(b)$  by  $\gamma(a)$ .
- **Page 188, line 4:** Replace  $J(q)$  by  $J(b)$ .
- **Page 188, last paragraph:** Replace  $b$  by  $a$  in each formula in this paragraph.
- **Page 189, last two displayed equations:** Replace  $b$  by  $a$  in three places.
- **Page 197, first line:** Insert the following just after “. . . local isometry.”: “Note that each line  $t \mapsto tX$  in  $T_pM$  is a  $\tilde{g}$ -geodesic, so  $(T_pM, \tilde{g})$  is complete by Corollary 6.14.”
- **Page 197, proof of Lemma 11.6, fourth line:** Change  $p \in M$  to  $p \in \pi(\widetilde{M})$ .
- **Page 197, line 4 from bottom:** After “ $M$  is complete,” insert “by Corollary 6.14.”
- **Page 203, last line:** Change “Walter” to “Wilhelm.”
- **Page 208, Problem 11-2(a):** Replace the last sentence by “If  $t_1 < t_2$  are zeros of  $v$ , then  $u$  must have at least one zero in  $(t_1, t_2)$ , unless  $a \equiv b$  on  $[t_1, t_2]$  and  $u$  and  $v$  are constant multiples of each other there.”
- **Page 208, Problem 11-3:** In the last sentence, insert “is” before “at least.”
- **Page 213:** The index entry for “Bianchi identity/contracted” should be page 125, not 124.
- **Page 216:** The index entry for “escape lemma” should be page 61, not 60.
- **Page 219:** Change “Klingenberg, Walter” to “Klingenberg, Wilhelm.”