Preface

Manifolds are everywhere. These generalizations of curves and surfaces to arbitrarily many dimensions provide the mathematical context for understanding "space" in all of its manifestations. Today, the tools of manifold theory are indispensable in most major subfields of pure mathematics, and outside of pure mathematics they are becoming increasingly important to scientists in such diverse fields as genetics, robotics, econometrics, computer graphics, biomedical imaging, and, of course, the undisputed leader among consumers (and inspirers) of mathematics—theoretical physics. No longer a specialized subject that is studied only by differential geometers, manifold theory is now one of the basic skills that all mathematics students should acquire as early as possible.

Over the past few centuries, mathematicians have developed a wondrous collection of conceptual machines designed to enable us to peer ever more deeply into the invisible world of geometry in higher dimensions. Once their operation is mastered, these powerful machines enable us to think geometrically about the 6-dimensional zero set of a polynomial in four complex variables, or the 10-dimensional manifold of 5×5 orthogonal matrices, as easily as we think about the familiar 2-dimensional sphere in $\mathbb{R}^3$. The price we pay for this power, however, is that the machines are built out of layer upon layer of abstract structure. Starting with the familiar raw materials of Euclidean spaces, linear algebra, and multivariable calculus, one must progress through topological spaces, smooth atlases, tangent bundles, cotangent bundles, immersed and embedded submanifolds, tensors, Riemannian metrics, differential forms, vector fields, flows, foliations, Lie derivatives, Lie groups, Lie algebras, and more—just to get to the point where one can even think about studying specialized applications of manifold theory such as gauge theory or symplectic topology.

This book is designed as a first-year graduate text on manifold theory, for students who already have a solid acquaintance with general topology, the fundamental group, and covering spaces, as well as basic undergraduate linear algebra and real analysis. The book is similar in philosophy and scope to the first volume of Spivak's classic text [Spi79], though perhaps a bit more dense. I have tried neither to write an encyclopedic introduction to manifold theory in its utmost generality, nor to write a simplified introduction that gives students a "feel" for the subject without the struggle that is required to master the tools. Instead, I have tried to find a middle path by introducing and using all of the standard tools of manifold theory, and proving all of its fundamental theorems, while avoiding unnecessary generalization or specialization. I try to keep the approach as concrete as possible, with pictures and intuitive discussions of how one should think geometrically about the abstract concepts, but without shying away from the powerful tools that modern mathematics has to offer. To fit in all of the basics and still maintain a reasonably sane pace, I have had to omit a number of important topics entirely, such as complex manifolds, infinite-dimensional manifolds, connections, geodesics, curvature, fiber bundles, sheaves, characteristic classes, and Hodge theory. Think of them as dessert, to be savored after completing this book as the main course.

The goal of my choice of topics is to cover those portions of smooth manifold theory that most people who will go on to use manifolds in mathematical or scientific research will need. To convey the book's compass, it is easiest to describe where it starts and where it ends.

The starting line is drawn just after topology: I assume that the reader has had a rigorous course in topology at the beginning graduate or advanced undergraduate level, including a treatment of the fundamental group and covering spaces. One convenient source for this material is my *Introduction to Topological Manifolds* [Lee00], which I wrote two years ago precisely with the intention of providing the necessary foundation for this book. There are other books that cover similar material well; I am especially fond of Sieradski's *An Introduction to Topology and Homotopy* [Sie92] and the new edition of Munkres's *Topology* [Mun00].

The finish line is drawn just after a broad and solid background has been established, but before getting into the more specialized aspects of any particular subject. For example, I introduce Riemannian metrics, but I do not go into connections or curvature. There are many Riemannian geometry books for the interested student to take up next, including one that I wrote five years ago [Lee97] with the goal of moving expediently in a one-quarter course from basic smooth manifold theory to some nontrivial geometric theorems about curvature and topology. For more ambitious readers, I recommend the beautiful recent books by Petersen [Pet98], Sharpe [Sha97], and Chavel [Cha93].

This subject is often called "differential geometry." I have deliberately avoided using that term to describe what this book is about, however, because the term applies more properly to the study of smooth manifolds endowed with some extra structure—such as Lie groups, Riemannian manifolds, symplectic manifolds, vector bundles, foliations—and of their properties that are invariant under structure-preserving maps. Although I do give all of these geometric structures their due (after all, smooth manifold theory is pretty sterile without some geometric applications), I felt that it was more honest not to suggest that the book is primarily about one or all of these geometries. Instead, it is about developing the general tools for working with smooth manifolds, so that the reader can go on to work in whatever field of differential geometry or its cousins he or she feels drawn to.

One way in which this emphasis makes itself felt is in the organization of the book. Instead of gathering the material about a geometric structure together in one place, I visit each structure repeatedly, each time delving as deeply as is practical with the tools that have been developed so far. Thus, for example, there are no chapters whose main subjects are Riemannian manifolds or symplectic manifolds. Instead, Riemannian metrics are introduced in Chapter 11 right after tensors; they then return to play major supporting roles in the chapters on orientations and integration, followed by cameo appearances in the chapters on de Rham cohomology and Lie derivatives. Similarly, symplectic structures make their first appearance at the end of the chapter on differential forms, and can be seen lurking in an occasional problem or two for a while, until they come into prominence at the end of the chapter on Lie derivatives. To be sure, there are two chapters (9 and 20) whose sole subject matter is Lie groups and/or Lie algebras, but my goals in these chapters are less to give a comprehensive introduction to Lie theory than to develop some of the more general tools that everyone who studies manifolds needs to use, and to demonstrate some of the amazing things one can do with those tools.

The book is organized roughly as follows. The twenty chapters fall into four major sections, characterized by the kinds of tools that are used.

The first major section comprises Chapters 1 through 6. In these chapters I develop as much of the theory of smooth manifolds as one can do using, essentially, only the tools of topology, linear algebra, and advanced calculus. I say "essentially" because, as the reader will soon find out, there are a great many definitions here that will be unfamiliar to most readers and will make the material seem very new. The reader's main job in these first six chapters is to absorb all the definitions and learn to think about familiar objects in new ways. It is the bane of this subject that there are so many definitions that must be piled on top of one another before anything interesting can be said, much less proved. I have tried, nonetheless, to bring in significant applications as early and as often as possible. By the end of these six chapters, the reader will have been introduced to topological manifolds, smooth manifolds, the tangent and cotangent bundles, and abstract vector bundles.

The next major section comprises Chapters 7 through 10. Here the main tools are the inverse function theorem and its corollaries. This is the first of four foundational theorems on which all of smooth manifold theory rests. It is applied primarily to the study of submanifolds (including Lie subgroups and vector subbundles), quotients of manifolds by group actions, embeddings of smooth manifolds into Euclidean spaces, and approximation of continuous maps by smooth ones.

The third major section, consisting of Chapters 11 through 16, uses tensors and tensor fields as its primary tools. Beginning with the definition (or, rather, two different definitions) of tensors, I introduce Riemannian metrics, differential forms, integration, Stokes's theorem (the second of the four foundational theorems), and de Rham cohomology. The section culminates in the de Rham theorem, which relates differential forms on a smooth manifold to its topology via its singular cohomology groups.

The last major section, Chapters 17 through 20, explores the circle of ideas surrounding integral curves and flows of vector fields, which are the smooth-manifold version of systems of ordinary differential equations. The main tool here is the fundamental theorem on flows, the third foundational theorem. It is a consequence of the basic existence, uniqueness, and smoothness theorem for ordinary differential equations. Both of these theorems are proved in Chapter 17. Flows are used to define Lie derivatives and describe some of their applications (most notably to symplectic geometry), to study tangent distributions and foliations, and to explore in some detail the relationship between Lie groups and their Lie algebras. Along the way, we meet the fourth foundational theorem, the Frobenius theorem, which is essentially a corollary of the inverse function theorem and the fundamental theorem on flows.

The Appendix (which most readers should read, or at least skim, first) contains a cursory summary of the prerequisite material on topology, linear algebra, and calculus that is used throughout the book. Although no student who has not seen this material before is going to learn it from reading the Appendix, I like having all of the background material collected in one place. Besides giving me a convenient way to refer to results that I want to assume as known, it also gives the reader a splendid opportunity to brush up on topics that were once (hopefully) well understood but may have faded a bit.

I should say something about my choices of conventions and notations. The old joke that "differential geometry is the study of properties that are invariant under change of notation" is funny primarily because it is alarmingly close to the truth. Every geometer has his or her favorite system of notation, and while the systems are all in some sense formally isomorphic, the transformations required to get from one to another are often not at all obvious to the student. Because one of my central goals is to prepare students to read advanced texts and research articles in differential geometry, I have tried to choose notation and conventions that are as close to the mainstream as I can make them without sacrificing too much internal consistency. When there are multiple conventions or notations in common use (such as the two common conventions for the wedge product or the Laplace operator), I explain what the alternatives are and alert the student to be aware of which convention is in use by any given writer. Striving for too much consistency in this subject can be a mistake, however, and I have eschewed absolute consistency whenever I felt it would get in the way of ease of understanding. I have also introduced some common shortcuts at an early stage, such as the Einstein summation convention and the systematic confounding of maps with their coordinate representations, both of which tend to drive students crazy at first, but pay off enormously in efficiency later.

This book has a rather large number of exercises and problems for the student to work out. Embedded in the text of each chapter are questions labeled as “exercises.” These are (mostly) short opportunities to fill in the gaps in the text. Many of them are routine verifications that would be tedious to write out in full, but are not quite trivial enough to warrant tossing off as obvious. I hope that conscientious readers will take the time at least to stop and convince themselves that they fully understand what is involved in doing each exercise, if not to write out a complete solution, because it will make their reading of the text far more fruitful. At the end of each chapter is a collection of (mostly) longer and harder questions labeled as “problems.” These are the ones from which I select written homework assignments when I teach this material, and many of them will take hours for students to work through. It is really only in doing these problems that one can hope to absorb this material deeply. I have tried insofar as possible to choose problems that are enlightening in some way and have interesting consequences in their own right. The results of many of them are used in the text.

I welcome corrections or suggestions from readers. I plan to keep an up-to-date list of corrections on my Web site, *www.math.washington.edu/~lee*. If that site becomes unavailable for any reason, the publisher will know where to find me.

Happy reading!

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Finally, I am deeply indebted to my beloved family—Pm, Nathan, and Jeremy—who once again have endured my preoccupation and extended absences with generosity and grace. This time I plan to thank them by not writing a book for a while.

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