Math 126 **Calculus With Analytic Geometry III** **Fall 1995**

FINAL EXAM ANSWERS

1. We have the following information on the function $f(x, y)$:

   - Among all directional derivatives of $f(x, y)$ at the point $P(2, -1)$, the one in the direction of $\vec{v} = (4, 3)$ is largest, and it is equal to 10.
   - $f(2, -1) = 3$.

   a) (10 points) Find the components of the gradient of $f(x, y)$ at the point $P(2, -1)$.

   grad $f =$ $\boxed{(8, 6)}$

   b) (5 points) Through the point $P(2, -1)$ passes a level curve of $f(x, y)$. The normal object to this level curve is (circle one):

   $\boxed{\textit{a line in the plane}}$ *a line in space*

   a plane in space *none of the above*

   c) (10 points) Write an equation for the normal object mentioned in (b).

   Normal line: $\boxed{(x, y) = (2 + 8t, -1 + 6t)}$

   Some of you were paying close enough attention to notice the typo above: the label on the answer box to part (c) gave away the answer the part (b)!

2. $f(x, y)$ is a function of two variables. We define a new function $g(u, v)$ by

$$g(u, v) = f(u^2 - v^2, u + v).$$

   a) (16 points) Express the partial derivatives g_u and g_v at an arbitrary point (u, v) in terms of u, v, and the partial derivatives f_x and f_y. Indicate where f_x and f_y are to be evaluated.

$$g_u = \boxed{f_x(u^2 - v^2, u + v)(2u) + f_y(u^2 - v^2, u + v)}$$

$$g_v = \boxed{f_x(u^2 - v^2, u + v)(-2v) + f_y(u^2 - v^2, u + v)}$$

   b) (9 points) Some level curves of $f(x, y)$ are pictured on the exam. Give the best approximation you can for the components of grad g at the point $(u, v) = (2, 1)$.

   This was the hardest problem on the test. From the answer to part (a) above, grad $g = (f_x(3, 3)(4) + f_y(3, 3), f_x(3, 3)(-2) + f_y(3, 3))$ *when* $(u, v) = (2, 1)$. *To evaluate this, you need to find* $(f_x, f_y) =$ grad f *at the point* $(x, y) = (3, 3)$. *This can be found from the graph. First, since* grad f *is normal to the level curves of* f, *it is parallel to* $(3.5, 1)$. *To find its length, approximate the rate of change of* f *along the direction* $(3.5, 1)$: $|\text{grad } f| \approx (1-0)/0.7 \approx 1.4$, *giving* grad $f \approx (1.4, 0.4)$. *Plugging this into the formula for* grad g, *we get:*

   grad $g =$ $\boxed{(6.0, -2.4) \text{ (approximately)}}$

3. $g(x,y,z)$ is a function of three variables. We do not know the formula for g, but we know the formula for the linear approximation of g at the point $B(2,2,2)$:

$$L(x,y,z) = 2x - 5y + z - 2.$$

If possible, find each of the following from the information given. If there is insufficient information, write IMPOSSIBLE.

a) (5 points) The gradient of g at the point $B(2,2,2)$.

grad $g =$ $\boxed{(2,-5,1)}$

b) (10 points) The directional derivative $D_{\vec{u}}g(2,2,2)$, where $\vec{u} = (3,4,0)$.

$D_{\vec{u}}g(2,2,2) =$ $\boxed{-\frac{14}{5}}$

c) (10 points) The normal line to the surface $g(x,y,z) = -6$ at the point $B(2,2,2)$.

Normal line: $\boxed{(x,y,z) = (2+2t, 2-5t, 2+t)}$

4. We are given the function $f(x,y) = \cos x + 2\sin y$, and M is the collection of all points (x,y,z) in space that satisfy the equation $z = f(x,y)$. The point $Q(0,\pi/2,3)$ is on M.

a) (10 points) Through Q there passes a tangent object of M. This tangent object is (read all and circle the best answer):

a curve in the plane *a curve in space*

a line in the plane *a line in space*

a curved surface in space $\boxed{\textit{a plane in space}}$

none of the above

b) (15 points) Write an equation for this tangent object.

Tangent object: $\boxed{z = 3}$

5. We are given the function $f(x,y) = x^3 - 6xy + y^3$. Find the coordinates of all critical points of $f(x,y)$, and use the second-derivative test to determine if possible the type of each critical point (local maximum, local minimum, saddle point). If the second-derivative test is inconclusive, write **???** in the 'Type' box.

Critical point:	$(0,0)$	Type:	Saddle point
Critical point:	$(2,2)$	Type:	Local minimum

6. The circle C shown on the exam is given by the equation $x^2 - 2x + y^2 = 0$.

a) (15 points) Using Lagrange multipliers, find all the critical points of the function $g(x, y) = x^2 - y^2$ **along the curve** C.

$\boxed{(\frac{1}{2}, \frac{\sqrt{3}}{2})}$ $\boxed{(\frac{1}{2}, -\frac{\sqrt{3}}{2})}$ $\boxed{(0, 0)}$ $\boxed{(2, 0)}$

b) (10 points) Find the global maximum and minimum values of $g(x, y)$ on C, and the coordinates of the points where they occur.

Global maximum: $\boxed{4}$ occurs at $(x, y) = \boxed{(2, 0)}$

Global minimum: $\boxed{-\frac{1}{2}}$ occurs at $(x, y) = \boxed{(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})}$

7. D is the filled-in region shown on the exam. The upper boundary curve of D is an arc of the circle with radius 5 and center at the origin. We are also given a function

$$f(x, y) = \frac{1}{1 + 2x + y}.$$

a) (10 points) Using the Basic Estimate, find constants k and K so that

$$k \leq \iint_D f(x, y)\, dA \leq K.$$

[Hint: use level curves to find the minimum and maximum values of the integrand on D.]

$$k = \boxed{\frac{25\pi}{8} \frac{1}{1 + 2\frac{5}{\sqrt{2}} + \frac{5}{\sqrt{2}}} \approx 0.85}$$

$$K = \boxed{\frac{25\pi}{8} \approx 9.8}$$

b) (15 points) Set up (but do not evaluate) the integral $\iint_D f(x, y)\, dA$ in polar coordinates.

$$\int_{\boxed{\theta = \frac{\pi}{4}}}^{\boxed{\theta = \frac{\pi}{2}}} \int_{\boxed{r = 0}}^{\boxed{r = 5}} \boxed{\frac{1}{1 + 2r\cos\theta + r\sin\theta}\, r}\; d\boxed{r}\; d\boxed{\theta}$$

8. D is the shaded parallelogram shown on the exam.

a) (15 points) Decide on an order of integration, and set up the double integral $\iint_D 2y\,dA$.

$$\int_{\boxed{x=0}}^{\boxed{x=3}} \int_{\boxed{y=3-x}}^{\boxed{y=4-x}} \boxed{2y}\, d\boxed{y}\, d\boxed{x}$$

b) (10 points) Evaluate the integral that you set up in (a).

$$\iint_D 2y\,dA = \boxed{12}$$