## Honors Calculus Handout 6: The Rational Numbers

## Some More Facts about Integers

An integer n is said to be *even* if it can be expressed in the form n = 2k for some integer k, and **odd** if it can be expressed as n = 2l + 1 for some integer l.

**Theorem 85.** Every integer is either even or odd, but not both.

*Proof.* First we'll show by induction that every positive integer is even or odd. For the base case, just note that 1 is odd because it can be written  $1 = 2 \cdot 0 + 1$ . For the inductive step, let n be a positive integer, and assume n is either even or odd. If n is even, then n = 2k for some integer k, and thus n + 1 = 2k + 1, which is odd. If n is odd, then n = 2l + 1 for some integer l, and then n + 1 = 2l + 2 = 2(l + 1), which is even. In either case, n + 1 is even or odd, and the induction is complete.

Now we'll show that all nonpositive integers are either even or odd. If n = 0, then  $n = 2 \cdot 0$ , which is even. And if n < 0, then -n is either even or odd by the argument above, so either -n = 2k or -n = 2l + 1. This implies n = 2(-k) in the first case, so n is even; and it implies n = -2l - 1 = 2(-l - 1) + 1 in the second, so n is odd. This completes the proof that every integer is even or odd.

To show that no integer can be both even and odd, suppose for contradiction that n is both. Then there are integers k and l such that n = 2k and n = 2l + 1. This means 2k = 2l + 1, which implies 1/2 = k + (-l), which is an integer. But this contradicts the fact that 1 is the smallest positive integer.  $\Box$ 

**Theorem 86.** [PROPERTIES OF EVEN AND ODD INTEGERS] Suppose m and n are integers.

- (a) m + n is even if and only if m and n are both even or both odd.
- (b) m + n is odd if and only if one of the summands is even and the other is odd.
- (c) mn is even if and only if m or n is even.
- (d) mn is odd if and only if m and n are both odd.
- (e)  $n^2$  is even if and only if n is even, and odd if and only if n is odd.

Proof. Exercise.

**Theorem 87.** The set  $\mathbb{Z}^+$  of positive integers has no upper bound in  $\mathbb{R}$ .

*Proof.* Suppose for the sake of contradiction that  $\mathbb{Z}^+$  has an upper bound. Since  $\mathbb{Z}^+$  is not empty, it has a least upper bound; let's call it M. Since M is the *least* upper bound, it follows that M - 1 is not an upper bound for  $\mathbb{Z}^+$ , or in other words, there exists some integer k such that k > M - 1. But this implies k + 1 > M, and since k + 1 is also a positive integer, this contradicts the fact that M is the least upper bound of  $\mathbb{Z}^+$ .

## **Rational Numbers**

A real number is called a *rational number* if it can be expressed in the form p/q, where p and q are integers and  $q \neq 0$ . The set of all rational numbers is denoted by  $\mathbb{Q}$ . A real number is said to be *irrational* if it is not rational.

**Theorem 88.** 0 and 1 are rational numbers.

*Proof.* Just note that 0 = 0/1 and 1 = 1/1.

**Theorem 89.** [CLOSURE OF  $\mathbb{Q}$ ] If a and b are rational numbers than so are a + b, a - b, and ab. If in addition  $b \neq 0$ , then a/b is a rational number.

*Proof.* These follow immediately from the formulas

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs}, \qquad \frac{p}{q} - \frac{r}{s} = \frac{ps - qr}{qs}, \qquad \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}, \qquad \frac{p}{q} \Big/ \frac{r}{s} = \frac{ps}{qr},$$

together with the facts that sums, differences, and products of integers are integers.

A fraction p/q is said to be in *lowest terms* if the largest integer that evenly divides both p and q is 1.

**Theorem 90.** Every rational number can be expressed as a fraction in lowest terms.

*Proof.* Suppose r is a rational number, and let S be the following set:

 $S = \{q \in \mathbb{Z}^+ : r \text{ has an expression of the form } p/q \text{ for some integer } p.$ 

The fact that r is rational means that S is nonempty, so by the well-ordering property of  $\mathbb{Z}^+$  (Theorem 79 on Handout 5), S contains a smallest positive integer; call it  $q_0$ . That means there is some integer  $p_0$  such that  $r = p_0/q_0$ .

We wish to show that  $p_0/q_0$  is in lowest terms. Suppose not: then there is an integer k > 1 that evenly divides both  $p_0$  and  $q_0$ . This means there are integers  $p_1$  and  $q_1$  such that  $p_0 = kp_1$  and  $q_0 = kq_1$ . Since k > 1 and  $q_0$  is positive, this implies  $q_1 = q_0/k$  is positive and strictly smaller than  $q_0$ . But then we have  $r = p_1/q_1$  with a denominator  $q_1$  smaller than  $q_0$ , which is a contradiction.

**Theorem 91.**  $\sqrt{2}$  is irrational.

*Proof.* Suppose for contradiction that  $\sqrt{2}$  is rational. By Theorem 90, we can write  $\sqrt{2} = p/q$  in lowest terms. Now this means  $p^2/q^2 = 2$ , or

$$p^2 = 2q^2.$$

By Theorem 86(e), this implies p is even, so p = 2k for some integer k. Substituting for p in the equation above, we conclude

$$(2p)^2 = 2q^2$$
, which implies  $2p^2 = q^2$ .

Using Theorem 86(e) once again, we conclude that q is also even. But this contradicts the fact that p/q is in lowest terms, so our assumption that  $\sqrt{2}$  is rational must have been false.

**Theorem 92.** [DENSITY OF RATIONAL NUMBERS] If a and b are real numbers such that a < b, then there exists a rational number c such that a < c < b.

*Proof.* By Theorem 87, there is a positive integer q such that q > 1/(b-a). Now consider the set S of all positive integers n such that n > aq. This set is nonempty by Theorem 87, so by the well-ordering property of  $\mathbb{Z}^+$ , S contains a smallest integer p. We will show that a < p/q < b.

First, the fact that  $p \in S$  means by definition that p > aq, and therefore a < p/q. To show that p/q < b, suppose for contradiction that  $p/q \ge b$ . We chose q such that q > 1/(b-a), which implies 1/q < b-a and therefore -1/q > a - b. Now consider the rational number (p-1)/q:

$$\frac{p-1}{q} = \frac{p}{q} + \frac{-1}{q} > b + (a-b) = a.$$

This in turn implies p-1 > aq, so  $p-1 \in S$ . But this contradicts our choice of p as the smallest number in S. This shows our assumption was false and therefore p/q < b.

**Theorem 93.** [DENSITY OF IRRATIONAL NUMBERS] If a and b are real numbers such that a < b, then there exists an irrational number c such that a < c < b.

*Proof.* Since  $a/\sqrt{2} < b/\sqrt{2}$ , the previous theorem implies that there is a rational number c such that  $a/\sqrt{2} < c < b/\sqrt{2}$ . If c = 0, then there is another rational c' such that  $a/\sqrt{2} < 0 < c' < b/\sqrt{2}$ , so after replacing c by c' if necessary, we may assume  $c \neq 0$ .

This in turn implies  $a < c\sqrt{2} < b$ . If  $c\sqrt{2}$  were rational, then  $\sqrt{2} = (c\sqrt{2})/c$  would also be rational, which is a contradiction; so  $d = c\sqrt{2}$  is an irrational number between a and b.