## Handout 6: The Rational Numbers

## Some More Facts about Integers

An integer $n$ is said to be even if it can be expressed in the form $n=2 k$ for some integer $k$, and odd if it can be expressed as $n=2 l+1$ for some integer $l$.

Theorem 85. Every integer is either even or odd, but not both.
Proof. First we'll show by induction that every positive integer is even or odd. For the base case, just note that 1 is odd because it can be written $1=2 \cdot 0+1$. For the inductive step, let $n$ be a positive integer, and assume $n$ is either even or odd. If $n$ is even, then $n=2 k$ for some integer $k$, and thus $n+1=2 k+1$, which is odd. If $n$ is odd, then $n=2 l+1$ for some integer $l$, and then $n+1=2 l+2=2(l+1)$, which is even. In either case, $n+1$ is even or odd, and the induction is complete.

Now we'll show that all nonpositive integers are either even or odd. If $n=0$, then $n=2 \cdot 0$, which is even. And if $n<0$, then $-n$ is either even or odd by the argument above, so either $-n=2 k$ or $-n=2 l+1$. This implies $n=2(-k)$ in the first case, so $n$ is even; and it implies $n=-2 l-1=2(-l-1)+1$ in the second, so $n$ is odd. This completes the proof that every integer is even or odd.

To show that no integer can be both even and odd, suppose for contradiction that $n$ is both. Then there are integers $k$ and $l$ such that $n=2 k$ and $n=2 l+1$. This means $2 k=2 l+1$, which implies $1 / 2=k+(-l)$, which is an integer. But this contradicts the fact that 1 is the smallest positive integer.

Theorem 86. [Properties of Even and Odd Integers] Suppose $m$ and $n$ are integers.
(a) $m+n$ is even if and only if $m$ and $n$ are both even or both odd.
(b) $m+n$ is odd if and only if one of the summands is even and the other is odd.
(c) $m n$ is even if and only if $m$ or $n$ is even.
(d) $m n$ is odd if and only if $m$ and $n$ are both odd.
(e) $n^{2}$ is even if and only if $n$ is even, and odd if and only if $n$ is odd.

Proof. Exercise.
Theorem 87. The set $\mathbb{Z}^{+}$of positive integers has no upper bound in $\mathbb{R}$.
Proof. Suppose for the sake of contradiction that $\mathbb{Z}^{+}$has an upper bound. Since $\mathbb{Z}^{+}$is not empty, it has a least upper bound; let's call it $M$. Since $M$ is the least upper bound, it follows that $M-1$ is not an upper bound for $\mathbb{Z}^{+}$, or in other words, there exists some integer $k$ such that $k>M-1$. But this implies $k+1>M$, and since $k+1$ is also a positive integer, this contradicts the fact that $M$ is the least upper bound of $\mathbb{Z}^{+}$.

## Rational Numbers

A real number is called a rational number if it can be expressed in the form $p / q$, where $p$ and $q$ are integers and $q \neq 0$. The set of all rational numbers is denoted by $\mathbb{Q}$. A real number is said to be irrational if it is not rational.

Theorem 88. 0 and 1 are rational numbers.
Proof. Just note that $0=0 / 1$ and $1=1 / 1$.
Theorem 89. [Closure of $\mathbb{Q}$ ] If $a$ and $b$ are rational numbers than so are $a+b, a-b$, and $a b$. If in addition $b \neq 0$, then $a / b$ is a rational number.

Proof. These follow immediately from the formulas

$$
\frac{p}{q}+\frac{r}{s}=\frac{p s+q r}{q s}, \quad \frac{p}{q}-\frac{r}{s}=\frac{p s-q r}{q s}, \quad \frac{p}{q} \cdot \frac{r}{s}=\frac{p r}{q s}, \quad \frac{p}{q} / \frac{r}{s}=\frac{p s}{q r}
$$

together with the facts that sums, differences, and products of integers are integers.
A fraction $p / q$ is said to be in lowest terms if the largest integer that evenly divides both $p$ and $q$ is 1 .
Theorem 90. Every rational number can be expressed as a fraction in lowest terms.
Proof. Suppose $r$ is a rational number, and let $S$ be the following set:

$$
S=\left\{q \in \mathbb{Z}^{+}: r \text { has an expression of the form } p / q \text { for some integer } p\right.
$$

The fact that $r$ is rational means that $S$ is nonempty, so by the well-ordering property of $\mathbb{Z}^{+}$(Theorem 79 on Handout 5), $S$ contains a smallest positive integer; call it $q_{0}$. That means there is some integer $p_{0}$ such that $r=p_{0} / q_{0}$.

We wish to show that $p_{0} / q_{0}$ is in lowest terms. Suppose not: then there is an integer $k>1$ that evenly divides both $p_{0}$ and $q_{0}$. This means there are integers $p_{1}$ and $q_{1}$ such that $p_{0}=k p_{1}$ and $q_{0}=k q_{1}$. Since $k>1$ and $q_{0}$ is positive, this implies $q_{1}=q_{0} / k$ is positive and strictly smaller than $q_{0}$. But then we have $r=p_{1} / q_{1}$ with a denominator $q_{1}$ smaller than $q_{0}$, which is a contradiction.
Theorem 91. $\sqrt{2}$ is irrational.
Proof. Suppose for contradiction that $\sqrt{2}$ is rational. By Theorem 90 , we can write $\sqrt{2}=p / q$ in lowest terms. Now this means $p^{2} / q^{2}=2$, or

$$
p^{2}=2 q^{2}
$$

By Theorem 86(e), this implies $p$ is even, so $p=2 k$ for some integer $k$. Subsituting for $p$ in the equation above, we conclude

$$
(2 p)^{2}=2 q^{2}, \quad \text { which implies } \quad 2 p^{2}=q^{2} .
$$

Using Theorem 86(e) once again, we conclude that $q$ is also even. But this contradicts the fact that $p / q$ is in lowest terms, so our assumption that $\sqrt{2}$ is rational must have been false.

Theorem 92. [Density of Rational Numbers] If $a$ and $b$ are real numbers such that $a<b$, then there exists a rational number $c$ such that $a<c<b$.

Proof. By Theorem 87, there is a positive integer $q$ such that $q>1 /(b-a)$. Now consider the set $S$ of all positive integers $n$ such that $n>a q$. This set is nonempty by Theorem 87 , so by the well-ordering property of $\mathbb{Z}^{+}, S$ contains a smallest integer $p$. We will show that $a<p / q<b$.

First, the fact that $p \in S$ means by definition that $p>a q$, and therefore $a<p / q$. To show that $p / q<b$, suppose for contradiction that $p / q \geq b$. We chose $q$ such that $q>1 /(b-a)$, which implies $1 / q<b-a$ and therefore $-1 / q>a-b$. Now consider the rational number $(p-1) / q$ :

$$
\frac{p-1}{q}=\frac{p}{q}+\frac{-1}{q}>b+(a-b)=a .
$$

This in turn implies $p-1>a q$, so $p-1 \in S$. But this contradicts our choice of $p$ as the smallest number in $S$. This shows our assumption was false and therefore $p / q<b$.

Theorem 93. [Density of Irrational Numbers] If $a$ and $b$ are real numbers such that $a<b$, then there exists an irrational number $c$ such that $a<c<b$.

Proof. Since $a / \sqrt{2}<b / \sqrt{2}$, the previous theorem implies that there is a rational number $c$ such that $a / \sqrt{2}<c<b / \sqrt{2}$. If $c=0$, then there is another rational $c^{\prime}$ such that $a / \sqrt{2}<0<c^{\prime}<b / \sqrt{2}$, so after replacing $c$ by $c^{\prime}$ if necessary, we may assume $c \neq 0$.

This in turn implies $a<c \sqrt{2}<b$. If $c \sqrt{2}$ were rational, then $\sqrt{2}=(c \sqrt{2}) / c$ would also be rational, which is a contradiction; so $d=c \sqrt{2}$ is an irrational number between $a$ and $b$.

