Math 134 **Honors Calculus** **Fall 2016**

Handout 8: The nth Root Function

In Exercise 30, Section 2.6, you proved the following theorem:

Theorem 107 (Existence and Uniqueness of nth Roots). *Let n be a positive integer.*

(a) *If a and b are real numbers such that $0 \leq a < b$, then $a^n < b^n$.*

(b) *Every nonnegative real number x has a unique nonnegative nth root $x^{1/n}$.*

In this note, we prove that the function $f(x) = x^{1/n}$ is continuous for all $x \geq 0$, and differentiable for all $x > 0$. First, a simple lemma. It is an analogue for the nth root function of part (a) above.

Lemma 108. *Let n be a positive integer. If a and b are real numbers such that $0 \leq a < b$, then $a^{1/n} < b^{1/n}$.*

Proof. Under the given hypotheses, there are three possibilities: $a^{1/n} > b^{1/n}$, $a^{1/n} = b^{1/n}$, or $a^{1/n} < b^{1/n}$. In the first case, Theorem 107(a) implies that $\left(a^{1/n}\right)^n > \left(b^{1/n}\right)^n$, or $a > b$, which contradicts our hypothesis. In the second case, substitution yields $\left(a^{1/n}\right)^n = \left(b^{1/n}\right)^n$, or $a = b$, which again is a contradiction. The only remaining possibility is $a^{1/n} < b^{1/n}$. □

Theorem 109 (Continuity of the nth Root Function). *The function $f(x) = x^{1/n}$ is continuous on the interval $[0, \infty)$.*

Proof. We need to show f is continuous from the right at $x = 0$, and continuous in the usual sense at $x = c$ for every $c > 0$. Let's first take care of the $x = 0$ case. Given $\varepsilon > 0$, set $\delta = \varepsilon^n$. Then if $0 \leq h < \delta$, Lemma 108 implies

$$|f(0+h) - f(0)| = |(0+h)^{1/n} - 0^{1/n}| = h^{1/n} < \delta^{1/n} = \varepsilon.$$

Now assume $c > 0$. Let $\varepsilon > 0$ be arbitrary. We're going to define three numbers $\delta_1, \delta_2, \delta_3$, and take δ to be the minimum of these three. First, in order for $f(c+h)$ to make sense, we have to restrict h to be small enough that $c + h > 0$, which we can do by setting $\delta_1 = c$ and requiring that $|h| < \delta_1$.

Let's do some computations. We need to find δ small enough so that $|h| < \delta$ implies $|f(c+h) - f(c)| < \varepsilon$, or equivalently $-\varepsilon < (c+h)^{1/n} - c^{1/n} < \varepsilon$. Working with the second of these inequalities, we find

$$\begin{aligned} (c+h)^{1/n} - c^{1/n} < \varepsilon \quad &\Leftrightarrow \quad (c+h)^{1/n} < c^{1/n} + \varepsilon \\ &\Leftrightarrow \quad c + h < \left(c^{1/n} + \varepsilon\right)^n \\ &\Leftrightarrow \quad h < \left(c^{1/n} + \varepsilon\right)^n - c, \end{aligned}$$

where the second line is justified by Theorem 107(a) in the $\Rightarrow$ direction and Lemma 110 in the $\Leftarrow$ direction. Based on this, we set $\delta_1 = \left(c^{1/n} + \varepsilon\right)^n - c$. Since $c^{1/n} + \varepsilon > c^{1/n}$, Theorem 107(a) implies that $\delta_1 > 0$. Similarly,

$$-\varepsilon < (c+h)^{1/n} - c^{1/n} \quad \Leftrightarrow \quad c^{1/n} - \varepsilon < (c+h)^{1/n}.$$

If $\varepsilon \geq c^{1/n}$, this is automatically satisfied because the left-hand side is nonpositive and the right-hand side is positive. But if $\varepsilon < c^{1/n}$, we can continue:

$$\begin{aligned} c^{1/n} - \varepsilon < (c+h)^{1/n} \quad &\Leftrightarrow \quad \left(c^{1/n} - \varepsilon\right)^n < c + h \\ &\Leftrightarrow \quad -\left(c - \left(c^{1/n} - \varepsilon\right)^n\right) < h. \end{aligned}$$

Based on this, we set $\delta_2 = c - \left(c^{1/n} - \varepsilon\right)^n$ if $\varepsilon < c^{1/n}$, and otherwise just choose $\delta_2 = \delta_1$ (or any positive number, really). Another argument based on Theorem 107(a) shows that $\delta_2 > 0$.

Now set $\delta = \min(\delta_1, \delta_2, \delta_3)$. Then $|h| < \delta$ implies $c + h > 0$, $h < \delta_1$, and $-\delta_2 < h$, and therefore the computations above yield

$$-\varepsilon < (c+h)^{1/n} - c^{1/n} < \varepsilon,$$

which is what we wanted to prove. □

For the proof of differentiability, we will need the following lemma.

Lemma 110. *If n is a nonnegative integer and a and b are real numbers, then*

$$(a-b)(a^n + a^{n-1}b + \cdots + ab^{n-1} + b^n) = a^{n+1} - b^{n+1}. \tag{1}$$

Proof. First we dispose of the special case $a = 0$, in which case both sides of the equation reduce to $-b^{n+1}$.

Next we'll prove the formula in the special case $a = 1$. Using summation notation, we can write the formula we have to prove as

$$(1-b)\left(\sum_{i=0}^{n} b^i\right) = 1 - b^{n+1}. \tag{2}$$

We proceed by induction on n. The base case is $n = 0$, in which case both sides of the equation reduce to $1 - b$.

Now let k be a nonnegative integer, and suppose (2) holds for $n = k$. Using the inductive hypothesis, we compute

$$\begin{aligned}
(1-b)\left(\sum_{i=0}^{k+1} b^i\right) &= (1-b)\left(\sum_{i=0}^{k} b^i + b^{k+1}\right) \\
&= (1-b)\left(\sum_{i=0}^{k} b^i\right) + (1-b)b^{k+1} \\
&= (1-b^{k+1}) + (b^{k+1} - b^{k+2}) \\
&= 1 - b^{k+2},
\end{aligned}$$

which is what we had to prove. This completes the $a = 1$ case.

Now suppose a is an arbitrary positive real number. Factoring out a^{n+1} and using the special case we just proved, we find that

$$\begin{aligned}
(a-b)(a^n + a^{n-1}b + \cdots + ab^{n-1} + b^n) &= a^{n+1}\left(1 - \frac{b}{a}\right)\left(1 + \left(\frac{b}{a}\right) + \dots \left(\frac{b}{a}\right)^n\right) \\
&= a^{n+1}\left(1 - \left(\frac{b}{a}\right)^{n+1}\right) \\
&= a^{n+1} - b^{n+1},
\end{aligned}$$

which completes the proof. □

Now comes the main theorem.

Theorem 111 (Differentiability of the nth Root Function). *The function $f(x) = x^{1/n}$ is differentiable for all $x > 0$, with derivative $f'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$.*

Proof. Suppose c is any positive real number; we will show that f is differentiable at $x = c$ with the derivative given by $f'(c) = \frac{1}{n}c^{\frac{1}{n}-1}$. Let us consider the difference quotient:

$$\frac{f(c+h) - f(c)}{h} = \frac{(c+h)^{1/n} - c^{1/n}}{h}.$$

To make the formulas simpler, we adopt the abbreviations $a = (c+h)^{1/n}$ and $b = c^{1/n}$, so that the difference quotient is $(a-b)/h$. Using Lemma 110 with $n-1$ in place of n, we find

$$\begin{aligned}
\frac{f(c+h) - f(c)}{h} &= \frac{a-b}{h}\left(\frac{a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}}{a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}}\right) \\
&= \frac{a^n - b^n}{h\left(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}\right)}.
\end{aligned}$$

Noting that $a^n - b^n = \big((c+h)^{1/n}\big)^n - \big(c^{1/n}\big)^n = (c+h) - c = h$, we can simplify this to

$$\frac{f(c+h) - f(c)}{h} = \frac{1}{a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}}.$$

By continuity of the nth root function, both $a = (c+h)^{1/n}$ and $b = c^{1/n}$ approach a limit of $c^{1/n}$ as $h \to 0$, and therefore the denominator approaches the limit $(c^{1/n})^{n-1} + (c^{1/n})^{n-1} + \cdots + (c^{1/n})^{n-1} = nc^{(n-1)/n}$. Since $c^{(n-1)/n} \neq 0$, we have

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \frac{1}{c^{(n-1)/n}}.$$

Simplification yields $f'(c) = \frac{1}{n} c^{\frac{1}{n}-1}$. □