Handout 4: Sets

All of mathematics uses set theory as an underlying foundation. Intuitively, a ***set*** is a collection of objects, considered as a whole. The objects that make up the set are called its ***elements*** or its ***members***. The elements of a set may be any objects whatsoever, but for our purposes, they will usually be mathematical objects such as numbers, functions, or other sets. The notation $x \in X$ means that the object x is an element of the set X. The words ***collection*** and ***family*** are synonyms for set.

In rigorous axiomatic developments of set theory, the words *set* and *element* are taken as primitive undefined terms. (It would be very difficult to *define* the word "set" without using some word such as "collection," which is essentially a synonym for "set.") Instead of giving a general mathematical definition of what it means to be a set, or for an object to be an element of a set, mathematicians characterize each particular set by giving a precise definition of what it means for an object to be a element of *that* set—this is called the set's ***membership criterion***. The membership criterion for a set X is a statement of the form "$x \in X \Leftrightarrow P(x)$," where $P(x)$ is some sentence that is true precisely for those objects x that are elements of X, and no others. For example, if $\mathbb{Q}$ is the set of all rational numbers, then the membership criterion for $\mathbb{Q}$ might be expressed as follows:

$$x \in \mathbb{Q} \quad \Leftrightarrow \quad x = p/q \text{ for some integers } p \text{ and } q \text{ with } q \neq 0.$$

The essential characteristic of sets is that two sets are equal if and only if they have the same elements. Thus if X and Y are sets, then $X = Y$ if and only if every element of X is an element of Y, and every element of Y is an element of X. Symbolically,

$$X = Y \qquad \text{if and only if} \qquad \forall x,\ x \in X \Leftrightarrow x \in Y.$$

If X and Y are sets such that every element of X is also an element of Y, then we say X is a ***subset of Y***, written $X \subseteq Y$. Thus

$$X \subseteq Y \qquad \text{if and only if} \qquad \forall x,\ x \in X \Rightarrow x \in Y.$$

The notation $Y \supseteq X$ ("Y is a ***superset of X***") means the same as $X \subseteq Y$. Using the concept of subsets, we can restate the criterion for two sets to be equal as follows:

$$X = Y \qquad \text{if and only if} \qquad X \subseteq Y \text{ and } Y \subseteq X.$$

If $X \subseteq Y$ but $X \neq Y$, we say that X is a ***proper subset of Y*** (or Y is a ***proper superset of X***). Some authors use the notations $X \subseteq Y$ and $Y \supseteq X$ to mean that X is a proper subset of Y; however, since other authors use the symbol "$\subseteq$" to mean *any* subset, not necessarily proper, we generally avoid using this notation, and instead say explicitly when a subset is proper.

Defining Sets

We already know about the set $\mathbb{R}$ of all real numbers, and its subset $\mathbb{R}^+$ of all positive real numbers. Most other sets we deal with will be built up from these in various ways. There are just a few ways to define new sets. You will see that in each case, the set is completely determined by its membership criterion.

- Defining a set by listing elements: Given any list of objects that can be explicitly named, the set containing those objects and no others is denoted by listing the objects between braces: $\{c_1, c_2, \dots, c_n\}$. The membership criterion is easy to express:

  $$a \in \{c_1, c_2, \dots, c_n\} \qquad \Leftrightarrow \qquad a = c_1 \text{ or } a = c_2 \text{ or } \dots \text{ or } a = c_n.$$

  For example, the set $\{0, 1, 2\}$ contains the numbers 0, 1, and 2, and nothing else. Because a set is completely determined by which elements it contains, it does not matter what order the elements are listed in or whether they are repeated; the notations $\{0, 1, 2\}$, $\{2, 1, 0\}$, and $\{0, 0, 1, 2, 1, 1\}$ all denote the same set. A set containing exactly one element, such as $\{1\}$, is called a ***singleton***. The set containing no elements at all is called the ***empty set***; it is usually denoted by $\varnothing$, but it can also be denoted by $\{\ \}$.

- Defining a set by specification: Given a set D and an open sentence $P(x)$ in which x represents an element of D, there is a set whose elements are precisely those $x \in D$ for which $P(x)$ is true. This set is denoted by either of the notations $\{x \in D : P(x)\}$ or $\{x \in D \mid P(x)\}$. (This notation is often called ***set-builder notation***.) Here is the membership criterion for this set:

$$a \in \{x \in D : P(x)\} \qquad \Leftrightarrow \qquad a \in D \text{ and } P(a).$$

  If the domain of x is understood, or is implicit in the condition $P(x)$, the same set can be denoted by $\{x : P(x)\}$. For example, the set of all positive real numbers can be described by either of the following notations:

$$\{x \in \mathbb{R} : x > 0\} \qquad \text{or} \qquad \{x : x \in \mathbb{R} \text{ and } x > 0\}.$$

  For some sets, there is a formula that represents a typical element of the set, as some variable or variables run through all elements of some predetermined domain. For example, the set of perfect squares can be described as the set of all numbers of the form n^2 as n runs through the integers. In this case, we often use the following variant of set-builder notation:

$$\{n^2 : n \in \mathbb{Z}\}.$$

  This is a shorthand notation for $\{x : x = n^2 \text{ for some } n \in \mathbb{Z}\}$.

- Special Notations for Sets of Real Numbers And Integers For intervals in the real numbers, we have interval notations such as (a, b) and $[0, \infty)$. There are eight types of intervals, defined on page 6 of the textbook.

  For integers, we typically use ellipses $(\ldots)$ to designate ranges of integers. The Here are the definitions

Operations on Sets

There are three important operations that can be used to combine sets to obtain other sets.

- Union: Given any sets X and Y, their ***union***, denoted by $X \cup Y$, is the set whose elements are all the objects that are elements of X or elements of Y (or both). The membership criterion is

$$x \in X \cup Y \qquad \text{if and only if} \qquad x \in X \text{ or } x \in Y.$$

  Unions of more than two sets are defined similarly:

$$x \in X_1 \cup \cdots \cup X_n \qquad \text{if and only if} \qquad x \in X_1 \text{ or } \ldots \text{ or } x \in X_n.$$

- Intersection: Given sets X and Y, their ***intersection***, denoted by $X \cap Y$, is the set whose elements are all the objects that are elements of both X and Y; thus

$$x \in X \cap Y \qquad \text{if and only if} \qquad x \in X \text{ and } x \in Y.$$

  Just as for unions, we can define intersections of more than two sets:

$$x \in X_1 \cap \cdots \cap X_n \qquad \text{if and only if} \qquad x \in X_1 \text{ and } \ldots \text{ and } x \in X_n.$$

  Given two sets X and Y, we say that ***X and Y intersect*** if $X \cap Y \neq \varnothing$, meaning that they have at least one element in common. We say that ***X and Y are disjoint*** if $X \cap Y = \varnothing$ (i.e., if they do not intersect), meaning that they have no elements in common. To say that more than two sets are disjoint means that each pair of sets are disjoint; in other words, there is no element that lies in more than one of the sets.

- Set difference: If X and Y are sets, their ***difference***, denoted by $X \smallsetminus Y$, is the set of all elements in X that are not in Y:

$$x \in X \smallsetminus Y \qquad \text{if and only if} \qquad x \in X \text{ and } x \notin Y.$$

Ordered Pairs and Cartesian Products

An ***ordered pair*** is a choice of two objects (which could be the same or different), called the ***components*** of the ordered pair, together with a specification of which is the first component and which is the second. The notation (a, b) means the ordered pair in which a is the first component and b is the second. The defining characteristic is that two ordered pairs are equal if and only if their first components are equal and their second components are equal:

$$(a, b) = (a', b') \qquad \text{if and only if} \qquad a = a' \text{ and } b = b'.$$

Notice that the ordered pair (a, b) is not the same as the *set* $\{a, b\}$, because the order of components matters in the former but not in the latter. Thus $(1, 2)$ and $(2, 1)$ are different ordered pairs, but $\{1, 2\}$ and $\{2, 1\}$ are the same set.

Given two sets X and Y, the set of all ordered pairs of the form (x, y) with $x \in X$ and $y \in Y$ is called the ***Cartesian product of X and Y***, and is denoted by $X \times Y$. Thus

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}.$$

More generally, for any positive integer n, an ***ordered n-tuple*** is a choice of n objects arranged in a sequence, denoted by $(a_1, \dots, a_n)$. The criterion for equality of ordered n-tuples is the obvious generalization of that for ordered pairs: two ordered n-tuples $(a_1, \dots, a_n)$ and $(a'_1, \dots, a'_n)$ are equal if and only if all of their corresponding components are equal:

$$(a_1, \dots, a_n) = (a'_1, \dots, a'_n) \qquad \text{if and only if} \qquad a_1 = a'_1,\ a_2 = a'_2,\ \dots,\ \text{and } a_n = a'_n.$$

If $X_1, \dots, X_n$ are sets, the notation $X_1 \times \cdots \times X_n$ denotes the set of all ordered n-tuples of the form $(x_1, \dots, x_n)$, in which $x_1 \in X_1$, $x_2 \in X_2$, $\dots$, and $x_n \in X_n$. In particular, if all of the sets are the same set X, the n-fold Cartesian product $X \times \cdots \times X$ is usually denoted by X^n; it is the set of all ordered n-tuples of the form $(x_1, \dots, x_n)$ in which $x_1, \dots, x_n$ are all elements of X. For example, $\mathbb{R}^n$ denotes the n-fold Cartesian product of $\mathbb{R}$ with itself; it is just the set of all ordered n-tuples $(x_1, \dots, x_n)$ in which each x_i is a real number.