Math 441 **Topology** **Autumn 2002**

BACKGROUND MATERIAL FROM ELEMENTARY REAL ANALYSIS

Reference: *Advanced Calculus, 3rd Edition*, by A. E. Taylor and W. R. Mann, New York, Wiley, 1983.

Throughout this course, we will use without proof (and usually without further comment) the following facts about the real numbers.

- The set of real numbers (denoted by $\mathbb{R}$) is endowed with two operations, addition and multiplication, that satisfy all of the usual rules of algebra, such as the commutative, associative, and distributive laws for addition and multiplication.
- $\mathbb{R}$ satisfies the *least upper bound property:* If S is a nonempty subset of $\mathbb{R}$ that has an upper bound, then S has a least upper bound. This number is also called the *supremum* of S and is denoted by $\sup S$ or $\operatorname{lub} S$. Similarly, if S is nonempty and bounded below, it has a greatest lower bound, called its *infimum* and denoted by $\inf S$ or $\operatorname{glb} S$.
- $\mathbb{R}$ has the following important subsets:
  - The set $\mathbb{N}$ of *natural numbers* is the smallest subset of $\mathbb{R}$ with the following properties: $1 \in \mathbb{N}$; and if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.
  - The set $\mathbb{Z}$ of *integers* consists of those real numbers m such that either $m = 0$, $m \in \mathbb{N}$, or $-m \in \mathbb{N}$.
  - The set $\mathbb{Q}$ of *rational numbers* consists of those real numbers x that can be expressed in the form $x = p/q$, where p and q are integers.
- If S is any subset of $\mathbb{N}$, then S has a smallest element.
- If a and b are real numbers, then $ab = 0$ if and only if $a = 0$ or $b = 0$.
- If a and b are real numbers, then exactly one of the following inequalities holds:

$$a < b, \quad a = b, \quad \text{or} \quad a > b.$$

- Inequalities satisfy the following properties:
  - If $a \leq b$ and $b \leq a$, then $a = b$.
  - If $a \leq b$ and $b \leq c$, then $a \leq c$. If in addition either $a < b$ or $b < c$, then $a < c$.
  - If $a > 0$ and $b > 0$, then $a + b > 0$ and $ab > 0$.
  - If $a < b$ and c is any real number, then $a + c < b + c$.
  - If $a < b$ and $c > 0$, then $ac < bc$.
  - If $a < b$ and $c < 0$, then $ac > bc$.
  - If a is any nonzero real number, then $a^2 > 0$.

- If $a \in \mathbb{R}$, the *absolute value* of a, denoted by $|a|$, is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

- Abolute values satisfy the following properties for all $a, b \in \mathbb{R}$:

$$\begin{aligned} |a| &= 0 \text{ if and only if } a = 0; \\ |a| &= |-a| \geq 0, \\ a &\leq |a|, \\ |ab| &= |a|\ |b|\,, \\ |a+b| &\leq |a| + |b|, \\ \big||a| - |b|\big| &\leq |a-b|. \end{aligned}$$

- An *interval* in $\mathbb{R}$ is a set of one of the following forms for $a, b \in \mathbb{R}$:

$$\begin{aligned} [a,b] &= \{x \in \mathbb{R} : a \leq x \leq b\}; \\ (a,b) &= \{x \in \mathbb{R} : a < x < b\}; \\ [a,b) &= \{x \in \mathbb{R} : a \leq x < b\}; \\ (a,b] &= \{x \in \mathbb{R} : a < x \leq b\}; \\ (a,\infty) &= \{x \in \mathbb{R} : a < x\}; \\ [a,\infty) &= \{x \in \mathbb{R} : a \leq x\}; \\ (-\infty,b) &= \{x \in \mathbb{R} : x < b\}; \\ (-\infty,b] &= \{x \in \mathbb{R} : x \leq b\}; \\ (-\infty,\infty) &= \mathbb{R}. \end{aligned}$$

- If $J \subset \mathbb{R}$ is an interval, $f\colon \mathbb{R} \to \mathbb{R}$ is a function, and $a \in J$, then we say f is *continuous at a* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in J$, $|x-a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$. If f is continous at every $a \in J$, then we say that f is *continuous on J* or simply *continuous*.

- If $\langle x_i \rangle$ is a sequence of real numbers and $x \in \mathbb{R}$, we say the sequence *converges to x* if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $i \geq N$ implies $|x_i - x| < \varepsilon$. In this case, we write $\langle x_i \rangle \to x$ or $\lim_{i\to\infty} x_i = x$.

- If a, b are real numbers with $a < b$ and $f\colon [a,b] \to \mathbb{R}$ is a continuous function, then the definite integral $\int_a^b f(x)\,dx$ is a well-defined real number.

- Suppose a and b are real numbers such that $a < b$, and $f, g\colon [a,b] \to \mathbb{R}$ are continuous functions.
  - $\displaystyle \left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$
  - If $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.