Math 441 **Topology** **Summer 2007**

Handout #2: Functions

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This handout replaces pp. 15–16 in the text.

1. Functions

We generally think of a function as a "rule," or some kind of "black box," that converts "inputs" to "outputs." Given an appropriate input object (typically a number), the function produces a well-defined output object that depends only on the input; applying the function again to the same input must always result in the same output.

But what *is* a function really? The words we used above to describe it ("rule" or "black box") are too vague to be useful as a mathematical definition. What kinds of rules might be acceptable to define functions? Do they have to be formulas? Do they have to be algorithms? Can a function be random?

Because it is extremely important to be able to treat a function as a well-defined mathematical object, we need an official definition that is unambiguous and as general as possible. Mathematicians have settled on the idea of defining a function as a certain kind of *set*.

First of all, before we define a function, we must decide on what kinds of objects will be acceptable as inputs and outputs. So as to allow maximum flexibility, we will require nothing special about the inputs and outputs other than that they come from specific sets.

Thus let A and B be arbitrary sets. We wish to define what we mean by "a function f from A to B." This means that every input will be an element of A, and every output will be an element of B. Whatever the "rule" may be that converts inputs to outputs, it has to be completely determined by knowing, for each input object $a \in A$, the unique object $b \in B$ that f assigns as the output.

Therefore, we make the following definition. If A and B are sets, a ***function from A to B*** is a subset $f \subset A \times B$ (i.e., a set of ordered pairs of the form (a, b) with $a \in A$ and $b \in B$), satisfying the following condition:

(1) For every $a \in A$, there is exactly one $b \in B$ such that $(a, b) \in f$.

It is sometimes useful to rephrase (1) as two separate conditions:

(2) For every $a \in A$, there is at least one $b \in B$ such that $(a, b) \in f$;

(3) For every $a \in A$, there is no more than one $b \in B$ such that $(a, b) \in f$.

Together, these two conditions (or, equivalently, the single condition (1)) ensure that each input $a \in A$ yields exactly one output $b \in B$. Condition (2) is often summarized by saying that f is ***everywhere defined*** (meaning that it gives an output for each object in A), and condition (3) by saying that f is ***uniquely defined*** (meaning that it cannot give two or more different outputs for the same input). If both conditions are satisfied, we say that f is ***well-defined***. Note that f being well-defined means nothing more nor less than "f is a function."

The set A is called the ***domain*** of f, and the set B is called its ***range*** or its ***codomain***. The statement "f is a function from A to B" is symbolized by

$$f\colon A \to B.$$

If a is any element of A, we let $f(a)$ denote the unique element $b \in B$ such that $(a, b) \in f$. Thus

$$b = f(a) \quad \Leftrightarrow \quad (a, b) \in f.$$

The element $f(a)$ is called the ***value of f at a***. Now that we have a clear understanding that a function is a particular kind of set, we will almost always dispense with the ordered pair notation and write $b = f(a)$, just as you have always done in calculus.

Here we need to make one very important observation: *The domain and codomain are part of the definition of a function.* Thus a function, properly speaking, is three sets: the domain A, the codomain B, and the set $f \subset A \times B$ of ordered pairs satisfying (1). To define a function, you need to give all three pieces of information. In order for two functions to be equal, they must satisfy three conditions:

- They have the same domain;
- They have the same codomain;
- For every element of the domain, they give the same value.

A typical way to define a function is to say "Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function such that $f(x) = x^3 + 1$." This means that f is a function whose domain is $\mathbb{R}$, whose codomain is $\mathbb{R}$, and whose value for each $x \in \mathbb{R}$ is the number given by $x^3 + 1$. Since this formula yields a unique element of $\mathbb{R}$ for every element $x \in \mathbb{R}$, the function f is well-defined. If there is no ambiguity about what $f(x)$ is supposed to be for each x, then nothing more needs to be said. But sometimes the definition needs to be checked to ensure that f is well-defined. For example, suppose we want to let $g\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$g(x) = \begin{cases} x, & x \geq 0, \\ -x, & x \leq 0. \end{cases}$$

(You probably recognize this as the absolute value function.) Then because we are giving two different formulas for $g(x)$ when $x = 0$, we need to verify that they both give the same value, so that g is well-defined.

There is another set associated with a function that must be clearly distinguished from the domain and codomain. Given a function $f\colon A \to B$, the ***image set of f*** (sometimes called just the ***image of f***) is the subset of B consisting of all elements that actually are values of f. It is denoted by $f(A)$. Symbolically,

$$f(A) = \{b \in B : b = f(a) \text{ for at least one } a \in A\}.$$

The image is a subset of the codomain, but it might or might not be equal to the codomain. For example, if $f\colon \mathbb{R} \to \mathbb{R}$ is the function defined by $f(x) = x^2$, then its codomain is $\mathbb{R}$, but its image is only the set $\bar{\mathbb{R}}^+$ of nonnegative real numbers. (Be warned that many mathematicians, particularly analysts, use the word *range* as a synonym for *image*. For this reason, I usually prefer using *codomain* instead of *range* to avoid ambiguity. But because Munkres uses *range*, you may use either *range* or *codomain* as you prefer. Just be sure to distinguish both terms from *image*.)