Math 442 **Differential Geometry** **Winter 2003**

Assignment #1: Due 1/13/03

I. Reading:

- Read the supplementary handout titled *Background Material on Linear Algebra and Real Analysis.*
- Read do Carmo, Appendix to Chapter 2, pages 118–127 (stop before Definition 1).
- Read do Carmo, Appendix to Chapter 5, pages 456–464 (through Proposition 9).
- Read do Carmo, Sections 1-1 through 1-4 (pages 1–14).

II. Practice problems:

1. Let $v, w \in \mathbb{R}^3$ be the following vectors:
$$v = (1,0,1), \quad w = (1,2,3).$$
Find all unit vectors that are orthogonal to both v and w.
2. Compute the determinant of each of the following matrices, and determine whether each matrix is singular or nonsingular:
$$A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 4 & 2 & 10 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & -3 & 2 \\ -1 & 2 & 1 \\ 3 & 1 & -1 \end{pmatrix}.$$
3. Compute the rank of each of the following matrices:
$$A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 4 & 2 & 10 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & -3 \\ -1 & 2 \\ 3 & 1 \end{pmatrix}.$$
4. do Carmo, Section 1-2 (p. 5), Exercise 1.

III. Required problems:

1. Let $V \subset \mathbb{R}^3$ be the subspace defined by the equation $3x - 2y = 0$. Find an orthonormal basis for V.
2. Let $S\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map whose matrix is $\left(\begin{smallmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \end{smallmatrix}\right)$. Find a basis for the nullspace of S.
3. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map whose matrix is
$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 0 & -1 \end{pmatrix}.$$
Find an orthonormal basis for the range of T.

4. For each of the following matrices, find all eigenvalues and their corresponding eigenspaces.

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} -2 & -1 & 0 \\ 0 & 1 & 1 \\ -2 & -2 & -1 \end{pmatrix}.$$

5. For each of the following functions $F\colon \mathbb{R}^2 \to \mathbb{R}^3$, compute the Jacobian matrix of F in terms of (u, v), and determine all points (u, v) at which the Jacobian has rank less than 2.
   (a) $F(u, v) = (uv, u^2 + v^2, u^2 - v^2)$.
   (b) $F(u, v) = (u, v, u^2 - v^2)$.
   (c) $F(u, v) = (\sin u \cos v, \sin u \sin v, \cos u)$.
6. Suppose $I = (a, b) \subset \mathbb{R}$ is an open interval, and $\boldsymbol{\alpha}\colon I \to \mathbb{R}^3$ and $\boldsymbol{\beta}\colon I \to \mathbb{R}^3$ are parametrized smooth curves. Show that the following identity holds for all $t \in I$:

$$\frac{d}{dt}\big(\boldsymbol{\alpha}(t) \cdot \boldsymbol{\beta}(t)\big) = \boldsymbol{\alpha}'(t) \cdot \boldsymbol{\beta}(t) + \boldsymbol{\alpha}(t) \cdot \boldsymbol{\beta}'(t).$$

7. do Carmo, Section 1-2 (page 5), Exercise 2.
8. do Carmo, Section 1-2 (page 5), Exercise 3.
9. do Carmo, Section 1-2 (page 5), Exercise 4.
10. do Carmo, Section 1-2 (page 5), Exercise 5.