Math 442 **Differential Geometry** **Winter 2003**

Assignment #4: Due 2/3/03

I. Reading:

- Read do Carmo, Section 2-3 and 2-4 (pages 69–88).

II. Practice problems:

1. do Carmo, Section 2-2 (pp. 65–69), #3, 7, 8, 9.

III. Required problems:

1. do Carmo, Section 2-2 (pp. 65–69), #4.
2. do Carmo, Section 2-2 (pp. 65–69), #11.
3. do Carmo, Section 2-2 (pp. 65–69), #12.
4. do Carmo, Section 2-2 (pp. 65–69), #13.
5. do Carmo, Section 2-2 (pp. 65–69), #15.
6. do Carmo, Section 2-2 (pp. 65–69), #16.
7. (a) For any vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, show that $|x_1| + \cdots + |x_n| \leq \sqrt{n}|x|$.

   (b) Suppose A is an $m \times n$ matrix all of whose entries satisfy $|A_{ij}| < C$ for some positive constant C. If x is any vector in $\mathbb{R}^m$, show that $|Ax| \leq C\sqrt{mn}|x|$.
8. Let $U \subset \mathbb{R}^n$ be an open set, let $F\colon U \to \mathbb{R}^m$ be a smooth map, and fix some point $p \in U$. In class (and in do Carmo), the differential dF_p is interpreted geometrically in terms of the action of F on tangent vectors to curves. This problem outlines another interpretation, as a linear approximation to the local behavior of F near p.

   For any $h \in \mathbb{R}^n$ small enough that $p + h \in U$, let us write

   $$\Delta F(h) = F(p+h) - F(p),$$
   $$E(h) = \Delta F(h) - dF_p(h).$$

   You should think of $E(h)$ as the error that we commit when we use $dF_p(h)$ as a linear approximation to $\Delta F(h)$.

   Theorem: *With U, F, p as above,*

   $$\lim_{h \to 0} \frac{|E(h)|}{|h|} = 0.$$

   (In other words, as h goes to zero, the error $E(h)$ can be made arbitrarily small in comparison with h.) Prove this theorem as follows.

   (a) Given $\varepsilon > 0$, show that there exists $\delta > 0$ such that $|h| < \delta$ implies that the entries of the matrix $dF_{p+h} - dF_p$ are all less than ε.

   (b) Suppose $m = 1$, so F is a scalar-valued function. By applying the mean-value theorem to the function $f(t) = F(p + th)$, show that $E(h) = dF_{p+th}(h) - dF_p(h)$ for some $t \in [0, 1]$.

   (c) Prove the theorem in the case $m = 1$.

   (d) Prove it for general m by applying the preceding result to each component function of F.