Math 442 **Differential Geometry** **Winter 2011**

Assignment #4: Due 2/4/11

Reading:

- Sections 3.1 & 3.2.

Written Assignment:

A. Bär, Exercise 3.1 (page 84).

B. Bär, Exercise 3.2 (page 86).

C. Bär, Exercise 3.3 (page 86).

D. The definition of a regular surface in $\mathbb{R}^3$ can also be adapted to curves in $\mathbb{R}^2$. A subset $C \subset \mathbb{R}^2$ is called a ***regular* 1*-manifold*** if it has the following property: for every point $p \in C$, there exists an open neighborhood V of p in $\mathbb{R}^2$, and in addition, there exists an open interval $I \subset \mathbb{R}$ and a regular parametrized curve $c\colon I \to \mathbb{R}^2$ such that $c(I) = C \cap V$ and $c\colon I \to C \cap V$ is a homeomorphism. (Many authors would call this a "regular curve in $\mathbb{R}^2$," but Bär has already appropriated "curve" to mean an equivalence class of parametrized curves.)

The following proposition is proved in exactly the same way as Proposition 3.1.6. You may use it without proof in the problems that follow.

Proposition D.1. *Let $V_0 \subset \mathbb{R}^2$ be open, and let $f\colon V_0 \to \mathbb{R}$ be a smooth function. Define a set $C \subset \mathbb{R}^2$ by*

$$C = \{(x, y) \in V_0 : f(x, y) = 0\}.$$

If $\operatorname{grad} f(p) \neq 0$ *for all* $p \in C$*, then* C *is a regular* 1*-manifold.*

Prove that each of the following subsets of $\mathbb{R}^2$ is a regular 1-manifold.

(a) The circle $\{(x, y) : x^2 + y^2 = r^2\}$ for any fixed choice of $r > 0$.

(b) The graph of any smooth function $f\colon \mathbb{R} \to \mathbb{R}$.

(c) The image of the logarithmic spiral of Example 2.1.6.

[Hint for (c): one way to proceed is to try to eliminate t and find a function f defined on some open set containing C that satisfies the conditions of Proposition D.1. Another is to prove that the parametrization given in Example 2.1.6 is a homeomorphism onto its image, by finding an expression for its inverse that is a continuous function of x and y.]

E. Let $H \subset \mathbb{R}^2$ be the right half-plane, defined by

$$H = \{(u, v) : u > 0\}.$$

Suppose $C \subset \mathbb{R}^2$ is a regular 1-manifold that is entirely contained in H. The ***surface of revolution determined by C*** is the set $S_C \subset \mathbb{R}^3$ defined by

$$S_C = \{(x, y, z) : (r(x, y), z) \in C\},$$

where $r(x, y) = \sqrt{x^2 + y^2}$. The 1-manifold C is called the ***generating curve*** for the surface. Show that S_C is a regular surface in $\mathbb{R}^3$. [Hint: construct local surface parametrizations of the form $F(t, \varphi) = (c^1(t) \cos \varphi, c^1(t) \sin \varphi, c^2(t))$.]

F. Define $T \subset \mathbb{R}^3$ by

$$T = \{(x, y, z) : (r(x, y) - 2)^2 + z^2 = 1\}, \qquad \text{where } r(x, y) = \sqrt{x^2 + y^2}.$$

Prove that T is a surface of revolution. Find the generating curve, and find four surface parametrizations that cover all of its points. Sketch T.