Math 442 | **Differential Geometry** | **Winter 2011**

Assignment #7: Due 2/25/11

Reading:

- Bär, Sections 3.5, 3.6, and the following parts of Section 3.8: Example 3.8.12 (p. 139) and subsection 3.8.3 (pp. 143–145).

Written Assignment:

Note: *In these problems, you'll need the following definitions. Suppose $S \subset \mathbb{R}^3$ is a regular surface, $p \in S$, W_p is the Weingarten map of S at p, and (w_i^j) is the matrix of W_p with respect to some basis of T_pS. The* ***Gauss curvature of S at p*** *is*

$$K(p) = \det(W_p) = w_1^1 w_2^2 - w_1^2 w_2^1,$$

and the ***mean curvature of S at p*** *is*

$$H(p) = \frac{1}{2}\operatorname{trace}(H_p) = \frac{1}{2}(w_1^1 + w_2^2).$$

A regular surface is called a ***minimal surface*** *if its mean curvature is identically zero. (We'll discuss where this name comes from later.)*

A. Let $U \subset \mathbb{R}^2$ be an open set, let $f\colon U \to \mathbb{R}$ be a smooth function, and let $S \subset \mathbb{R}^3$ be the graph of f. Compute the matrix of the second fundamental form of S with respect to the parametrization of Example 3.1.4 (p. 82). [You may quote the results of previously assigned homework problems.]

B. Bär, Exercise 3.20 (p. 139).

C. Let $H = \{(u,v) : u > 0\}$ be the right half-plane, and let $C \subset H$ be a regular 1-manifold. Suppose for simplicity that C has a global parametrization: in other words, C is the image of a regular parametrized curve $c\colon I \to H$, with component functions $c(t) = (r(t), s(t))$ (so $r(t) > 0$ for all $t \in I$). Let S_C denote the surface of revolution determined by C as in Assignment 4. For all of the following problems, use a parametrization of the form $F(t,\varphi) = (r(t)\cos\varphi, r(t)\sin\varphi, s(t))$ with (t,φ) restricted to a suitable domain (which you don't have to specify explicitly).

   (a) Show that the following formula gives a unit normal field on the image of F:

   $$N(F(t,\varphi)) = \frac{1}{\sqrt{\dot r(t)^2 + \dot s(t)^2}}\big(\dot s(t)\cos\varphi,\ \dot s(t)\sin\varphi,\ -\dot r(t)\big).$$

   (b) Using the basis for T_pS_C determined by F and the normal field N defined above, compute the matrices of the first fundamental form, second fundamental form, and Weingarten map at an arbitrary point $p = F(t,\varphi) \in S$ in terms of $r(t)$, $s(t)$, and their derivatives. Verify that your formulas match the computations on pp. 143–144 of the textbook in the special case in which $s(t) \equiv t$.

   (c) Now assume that c is unit-speed, and prove that the Gauss curvature at any point $p = F(s,t) \in S$ can be expressed as

   $$K(p) = -\frac{\ddot r(t)}{r(t)}.$$

   [Hint: first prove that $\ddot r(t)\dot r(t) + \ddot s(t)\dot s(t) \equiv 0$.]

   (d) Prove that the Gauss curvature of S is everywhere zero if and only if C is contained in a straight line. [Remark: this shows that the only surfaces of revolution with zero Gauss curvature are portions of planes, cones, and cylinders.]

D. Bär, Exercise 3.31 (p. 145). [Hint: personally, I find the hint in the back of the book confusing. My suggestion is just to use the formulas you derived in the preceding problem for a surface of revolution generated by a non-unit-speed curve. The identities $\cos t = \cos^2(t/2) - \sin^2(t/2)$ and $\sin t = 2\sin(t/2)\cos(t/2)$ will probably be useful.]