Math 442 | **Differential Geometry** | **Winter 2012**

Assignment #1 (CORRECTED): Due 1/13/12

Reading:

- Read the handout titled "Background Material from Multivariable Calculus."
- Read Sections 1.1 and 1.2 (in Montiel and Ros).
- Skim Section 1.3.

Written Assignment:

A. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^k$ is a linear map. Show that f is its own differential at each point: more precisely, for every $p \in \mathbb{R}^n$, show that the linear map $(df)_p$ is equal to f itself.

B. Now suppose $f\colon \mathbb{R}^n \to \mathbb{R}^k$ is an affine map, written as $f(x) = Ax + b$ for some linear map A and some $b \in \mathbb{R}^k$. Show that for each $p \in \mathbb{R}^n$, $(df)_p = A$.

C. Suppose $U \subset \mathbb{R}^n$ is an open set, $f\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth function, and $p \in U$. Let $\Delta f\colon \mathbb{R}^n \to \mathbb{R}$ be defined by $\Delta f(v) = f(p+v) - f(p)$. (This formula was written incorrectly in the middle of page 2 of the handout.) Prove that

$$\lim_{v \to 0} \frac{|\Delta f(v) - (df)_p(v)|}{|v|} = 0.$$

(Remark: this says that the differential is a good linear approximation to Δf, in the sense that the error can be made as small as we want compared to $|v|$ by taking v sufficiently small.) [Hint: apply the mean value theorem to the function $g\colon [0,1] \to \mathbb{R}$ defined by $g(t) = f(p + tv)$, and use the continuity of $\partial f / \partial x_i$ and the definition of the limit.]

D. Show that a composition of two rigid motions of $\mathbb{R}^n$ is again a rigid motion, and the inverse of a rigid motion is a rigid motion.

E. Recall that an ***orthogonal matrix*** is an $n \times n$ matrix A whose associated linear map $A\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies $(Ax) \cdot (Ay) = x \cdot y$ for all $x, y \in \mathbb{R}^n$. Show that a square matrix is orthogonal matrix if and only if its columns are orthonormal vectors. (A set of vectors $\{v_1, \dots, v_k\}$ in $\mathbb{R}^n$ is said to be ***orthonormal*** if $|v_i| = 1$ for each i and $v_i \cdot v_j = 0$ for $i \neq j$. The dot represents the Euclidean dot product, which Montiel & Ros write with angle brackets: $\langle f(t), g(t) \rangle = f(t) \cdot g(t)$.)

F. Suppose $I \subset \mathbb{R}$ is an open interval and $\alpha, \beta\colon I \to \mathbb{R}^n$ are smooth curves. Prove that

$$\frac{d}{dt}\big(\alpha(t) \cdot \beta(t)\big) = \alpha'(t) \cdot \beta(t) + \alpha(t) \cdot \beta'(t).$$

G. Suppose $\alpha\colon I \to \mathbb{R}^n$ is a smooth curve whose distance from the origin is constant: $|\alpha(t)| \equiv c$ for all $t \in I$. Prove that $\alpha'(t) \perp \alpha(t)$ for all t.