Differentials of Surface Maps

This handout should be read in place of Section 3.4 of the textbook.

The Differential of a Map Out of a Surface

In Handout 1 from last quarter, we defined the differential of a smooth map at a point: it's the linear map whose matrix is the Jacobian matrix of the function. Theorem 1.4 showed how the differential can be interpreted as a "best linear approximation" to the map near the given point.

Now we would like to define a differential of a map between surfaces. The problem, however, is that there is no way to take partial derivatives of such a map, so we can't use the Jacobian matrix. We have to come up with a different definition.

We begin with the somewhat simpler case of smooth maps from surfaces into Euclidean spaces. The key to making sense of the differential of such a map is the chain rule. Suppose $\Omega \subseteq \mathbb{R}^n$ is an open subset, $F\colon \Omega \to \mathbb{R}^k$ is a smooth map, and $p \in \Omega$. Given a vector $w \in \mathbb{R}^n$, we can always find a smooth curve $\sigma\colon (-\varepsilon, \varepsilon) \to \Omega$ that satisfies $\sigma(0) = p$ and $\sigma'(0) = w$. (One obvious example of such a curve is the straight line parametrized by $\sigma(t) = p + tw$; but there are many other possibilities.) The composite map $F \circ \sigma\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^k$ is a curve in $\mathbb{R}^k$, and the chain rule tells us that its velocity is given by

$$(F \circ \sigma)'(0) = dF_p(\sigma'(0)) = dF_p(w).$$

If we're careful, we can make sense of the left-hand side of this formula for a function defined only on a surface, and thus we can use it as a *definition* of the right-hand side.

Here is the official definition. Suppose $S \subseteq \mathbb{R}^3$ is a regular surface and $F\colon S \to \mathbb{R}^k$ is a smooth map. For any $p \in S$, we define the ***differential of F at p*** to be the map $dF_p\colon T_pS \to \mathbb{R}^k$ given by

$$dF_p(w) = (F \circ \sigma)'(0), \tag{2.1}$$

where $\sigma\colon (-\varepsilon, \varepsilon) \to S$ is any smooth curve in S satisfying $\sigma(0) = p$ and $\sigma'(0) = w$. (Such a curve exists by definition of T_pS.) The next theorem shows that dF_p is a well-defined linear map.

Theorem 2.1 (Properties of the Differential). *Let $S \subseteq \mathbb{R}^3$ be a smooth surface, $F\colon S \to \mathbb{R}^k$ be a smooth map, and $p \in S$.*

(a) *For each $w \in T_pS$, the vector $dF_p(w) \in \mathbb{R}^k$ is well defined, independently of the choice of σ.*
(b) *The map $dF_p\colon T_pS \to \mathbb{R}^k$ so defined is linear.*
(c) *If W is an open neighborhood of p in $\mathbb{R}^3$ and $\widetilde{F}\colon W \to \mathbb{R}^k$ is a smooth map whose restriction to $S \cap W$ is equal to $F|_{S\cap W}$, then*

$$dF_p = d\widetilde{F}_p\Big|_{T_pS}. \tag{2.2}$$

Proof. Let $w \in T_pS$ be arbitrary, and let $\sigma\colon (-\varepsilon, \varepsilon) \to S$ be a curve satisfying $\sigma(0) = p$ and $\sigma'(0) = w$. By Proposition 3.2.12 (applied to the component functions of F), there exist an open neighborhood W of p in $\mathbb{R}^3$ and a smooth map $\widetilde{F}\colon W \to \mathbb{R}^k$ whose restriction to $S \cap W$ is equal to F. Because $\widetilde{F}$ is defined on a Euclidean open set, its differential is defined in the usual way (via the

Jacobian matrix), and we can apply the chain rule to it. Using the fact that (if ε is small enough) σ takes its values in $S \cap W$, where $\widetilde{F} = F$, we compute

$$\begin{aligned}(F \circ \sigma)'(0) &= (\widetilde{F} \circ \sigma)'(0) \\ &= d\widetilde{F}_p\big(\sigma'(0)\big) \\ &= d\widetilde{F}_p(w).\end{aligned}$$

This shows, first, that $dF_p(w) = (F \circ \sigma)'(0)$ is well defined, because $d\widetilde{F}_p(w)$ does not depend on the choice of σ at all. Second, it proves part (c) of the theorem, because this same computation applies to any smooth extension of F defined on an open neighborhood of p. And finally, it proves part (b), because $d\widetilde{F}_p\colon \mathbb{R}^3 \to \mathbb{R}^k$ is linear and the restriction of a linear map to a linear subspace is also linear. □

Thanks to this theorem, we have two effective ways to compute the differential of a function whose domain is a regular surface: either by choosing an appropriate smooth curve and using (2.1), or by choosing a smooth extension and using (2.2). Although either of these methods requires making a choice, the theorem guarantees that the result will be independent of choices.

Maps Between Surfaces

Next, we will see how to make sense of the differential of a map between surfaces. Given a smooth map $F\colon S_1 \to S_2$ between regular surfaces, we can think of F as a map into $\mathbb{R}^3$, and then the preceding theorem shows that for each $p \in S_1$ we have a well-defined linear map $dF_p\colon T_pS_1 \to \mathbb{R}^3$. The next proposition shows that it actually takes its values in $T_{F(p)}S_2$.

Proposition 2.2 (Differential of a Map Between Surfaces). *Suppose S_1 and S_2 are smooth surfaces and $F\colon S_1 \to S_2$ is a smooth map. For each $p \in S_1$, the differential dF_p maps T_pS_1 into $T_{F(p)}S_2$.*

Proof. Let $F\colon S_1 \to S_2$ be a smooth map, and let p be an arbitrary point of S_1. Given $w \in T_pS_1$, we can find a smooth curve $\sigma\colon (-\varepsilon, \varepsilon) \to S_1$ such that $\sigma(0) = p$ and $\sigma'(0) = w$. Since F takes its values in S_2, it follows that $F \circ \sigma$ is a smooth curve in S_2 satisfying $(F \circ \sigma)(0) = F(\sigma(0)) = F(p)$. Thus, by the very definition of the tangent plane, its initial velocity $(F \circ \sigma)'(0)$ is an element of $T_{F(p)}S_2$. But the definition of the differential shows that $dF_p(w) = (F \circ \sigma)'(0)$, so $dF_p(w)$ is an element of $T_{F(p)}S_2$ as claimed. □

As in the case of maps from a surface to $\mathbb{R}^k$, the differential of a map between surfaces can also be computed in two ways: either by choosing an appropriate smooth curve or by choosing a smooth extension. Here are some simple examples of differentials between surfaces that can be computed using smooth extensions.

Example 2.3 (Differentials).

(a) If $F\colon S_1 \to S_2$ is a smooth map that is the restriction of a linear map $A\colon \mathbb{R}^3 \to \mathbb{R}^3$, then for each $p \in S_1$, the differential dF_p is equal to the restriction of dA_p, which in turn is equal to A itself. Thus $dF_p(w) = Aw$ for every $w \in T_pS_1$, or equivalently $dF_p = A|_{T_pS_1}$.

(b) Suppose $S \subseteq \mathbb{R}^3$ is a regular surface and $\mathrm{id}_S\colon S \to S$ is the identity map. Because id_S is the restriction of $\mathrm{id}_{\mathbb{R}^3}$, which is linear, the preceding argument shows that for each $p \in S$, the differential $d(\mathrm{id}_S)_p$ is equal to the identity map of T_pS.

You will have opportunities to practice more such computations on the next homework assignment.

The next few theorems show that differentials of maps between surfaces behave similarly to ordinary differentials.

Theorem 2.4 (Chain Rule for Surface Maps). *Suppose $S_1, S_2, S_3 \subseteq \mathbb{R}^3$ are regular surfaces, and $F\colon S_1 \to S_2$ and $G\colon S_2 \to S_3$ are smooth maps. For any $p \in S_1$,*

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p. \tag{2.3}$$

The same is true if one or more of the surfaces S_1, S_2, S_3 are replaced by open subsets of Euclidean spaces.

Proof. First consider the case in which S_1, S_2, and S_3 are all regular surfaces. Given $p \in S_1$, Proposition 3.2.12 shows that we can extend F to a smooth map $\widetilde{F}\colon W_1 \to \mathbb{R}^3$ defined on a neighborhood W_1 of p in $\mathbb{R}^3$, and we can extend G to a smooth map $\widetilde{G}\colon W_2 \to \mathbb{R}^3$ defined on a neighborhood W_2 of $F(p)$ in $\mathbb{R}^3$. The composition $\widetilde{G} \circ \widetilde{F}$ is defined and smooth on the set $W_1 \cap \widetilde{F}^{-1}(W_2)$, which is a neighborhood of p. The ordinary chain rule (Theorem 1.2 on Handout 1) shows that

$$d\big(\widetilde{G} \circ \widetilde{F}\big)_p = d\widetilde{G}_{\widetilde{F}(p)} \circ d\widetilde{F}_p.$$

Then Theorem 2.1(c) (together with the fact that $\widetilde{F}(p) = F(p)$) shows that (2.3) follows from this.

If one or more of the surfaces are replaced by Euclidean open sets, the argument is even simpler, because we don't need to find extensions for the maps whose domains are open subsets of Euclidean spaces. □

Theorem 2.5. *If $F\colon S_1 \to S_2$ is a diffeomorphism, then for each $p \in S_1$, the linear map $dF_p\colon T_pS_1 \to T_{F(p)}S_2$ is invertible, with inverse given by*

$$\big(dF_p\big)^{-1} = d\big(F^{-1}\big)_{F(p)}.$$

Proof. Let $p \in S_1$ be arbitrary. Because $F^{-1} \circ F = \mathrm{id}_{S_1}$ and $F \circ F^{-1} = \mathrm{id}_{S_2}$, the chain rule implies

$$\begin{aligned} d\big(F^{-1}\big)_{F(p)} \circ dF_p &= d\big(F^{-1} \circ F\big)_p = d(\mathrm{id}_{S_1})_p = \mathrm{id}_{T_pS_1}, \\ dF_p \circ d\big(F^{-1}\big)_{F(p)} &= d\big(F \circ F^{-1}\big)_{F(p)} = d(\mathrm{id}_{S_2})_{F(p)} = \mathrm{id}_{T_{F(p)}S_2}. \end{aligned}$$

This shows that $d(F^{-1})_{F(p)}$ is a two-sided inverse for dF_p, and therefore it is its inverse. □

The converse to the preceding theorem is not true in general; a smooth map between surfaces can have invertible differential everywhere without being injective or surjective. However, the following theorem is a *local* converse. (It is Corollary 3.4.28 in the textbook.)

Theorem 2.6 (Inverse Function Theorem for Surfaces). *Suppose $S_1, S_2 \subseteq \mathbb{R}^3$ are regular surfaces, $F\colon S_1 \to S_2$ is a smooth map, and $p \in S_1$ is a point such that dF_p is invertible. Then there exist relative neighborhoods V_1 of p in S_1 and V_2 of $F(p)$ in S_2 such that $F|_{V_1}\colon V_1 \to V_2$ is a diffeomorphism.*

Proof. The fact that F is smooth means that there are local parametrizations $\varphi\colon U_1 \to S_1$ and $\psi\colon U_2 \to S_2$ such that $p \in \varphi(U_1)$, $F(\varphi(U_1)) \subseteq \psi(U_2)$, and $\psi^{-1} \circ F \circ \varphi$ is smooth in the ordinary sense. Let $\widehat{F} = \psi^{-1} \circ F \circ \varphi$, which is a smooth map from U_1 to U_2. Let $a_0 = \varphi^{-1}(p) \in U_1$ and

$b_0 = \psi^{-1}(F(p)) \in U_2$. Note that both φ and ψ are diffeomorphisms if we think of U_1 and U_2 as surfaces in the xy plane as in Example 3.2.8. Thus by the chain rule for surfaces,

$$d\widehat{F}_{a_0} = d\big(\psi^{-1} \circ F \circ \varphi\big)_{a_0} = d\big(\psi^{-1}\big)_{F(p)} \circ dF_p \circ d\varphi_{a_0}. \tag{2.4}$$

All three linear maps on the right-hand side are isomorphisms, and therefore so is the linear map on the left-hand side. The ordinary inverse function theorem (Theorem 1.7 in Handout 1) applied to $\widehat{F}$ shows that there are neighborhoods A_0 of a_0 and B_0 of b_0 such that $\widehat{F}$ restricts to a diffeomorphism from A_0 to B_0. Then the restriction of F to $\varphi(A_0)$ can be written as the following composition of diffeomorphisms:

$$F|_{\varphi(A_0)} = \psi \circ \widehat{F} \circ \varphi^{-1}|_{\varphi(A_0)},$$

and thus it is a diffeomorphism from $\varphi(A_0)$ to $F(\varphi(A_0))$. □

The expression (2.4) also provides a useful method for computing the differential in terms of parametrizations. As a linear map, dF_p is completely determined by what it does to the elements of any basis of T_pS. Since a local parametrization determines a convenient basis, it is useful to see how to compute the differential in terms of such a basis.

Example 2.7. Suppose $F\colon S_1 \to S_2$ is a smooth map between regular surfaces. Let $p \in S_1$ be arbitrary, and let $\varphi\colon U_1 \to S_1$ and $\psi\colon U_2 \to S_2$ be local parametrizations such that $p \in \varphi(U_1)$ and $F(\varphi(U_1)) \subseteq \psi(U_2)$, and let $\widehat{F} = \psi^{-1} \circ F \circ \varphi\colon U_1 \to U_2$ as above. (The map $\widehat{F}$ is sometimes called the ***coordinate representation of F***.) The vectors $\{d\varphi_{a_0}(\mathbf{e}_1),\ d\varphi_{a_0}(\mathbf{e}_2)\}$ form a basis for T_pS_1; let us abbreviate them as $\varphi_u = d\varphi_{a_0}(\mathbf{e}_1)$ and $\varphi_v = d\varphi_{a_0}(\mathbf{e}_2)$. Formula (2.4) implies

$$dF_p \circ d\varphi_{a_0} = d\psi_{b_0} \circ d\widehat{F}_{a_0},$$

and therefore we can compute the action of dF_p on φ_u as follows:

$$dF_p(\varphi_u) = dF_p \circ d\varphi_{a_0}(\mathbf{e}_1) = d\psi_{b_0} \circ d\widehat{F}_{a_0}(\mathbf{e}_1).$$

On the other hand, since $\widehat{F}$ is a smooth map between Euclidean open sets, its differential is given by its Jacobian matrix, so the right-hand side above can be expanded as follows:

$$\begin{aligned} dF_p(\varphi_u) &= d\psi_{b_0}\left(\frac{\partial \widehat{F}_1}{\partial u}(a_0)\mathbf{e}_1 + \frac{\partial \widehat{F}_2}{\partial u}(a_0)\mathbf{e}_2\right) \\ &= \frac{\partial \widehat{F}_1}{\partial u}(a_0)\psi_u + \frac{\partial \widehat{F}_2}{\partial u}(a_0)\psi_v, \end{aligned}$$

where we use the abbreviations $\psi_u = d\psi_{b_0}(\mathbf{e}_1)$ and $\psi_v = d\psi_{b_0}(\mathbf{e}_2)$. A similar computation shows

$$dF_p(\varphi_v) = \frac{\partial \widehat{F}_1}{\partial v}(a_0)\psi_u + \frac{\partial \widehat{F}_2}{\partial v}(a_0)\psi_v.$$

By linearity, this means that the result of applying dF_p to an arbitrary vector $A\varphi_u + B\varphi_v$ is $C\psi_u + D\psi_v$, where the coefficients C and D are determined by matrix multiplication as follows:

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \widehat{F}_1}{\partial u}(a_0) & \dfrac{\partial \widehat{F}_1}{\partial v}(a_0) \\ \dfrac{\partial \widehat{F}_2}{\partial u}(a_0) & \dfrac{\partial \widehat{F}_2}{\partial v}(a_0) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

In other words, in terms of the bases $\{\varphi_u, \varphi_v\}$ for T_pS_1 and $\{\psi_u, \psi_v\}$ for $T_{F(p)}S_2$, the differential dF_p is represented by the Jacobian matrix of the function $\widehat{F}$.