Math 443 | **Differential Geometry** | **Spring 2013**

Differential Geometry

Assignment #1
Due 4/12/13

Reading:

- Handout 2 (*Differentials of Surface Maps*)
- Handout 3 (*Bilinear and Quadratic Forms*)

Written Problems:

(1) [AT] Exercise 3.19.

(2) [AT] Exercise 3.31. (For this problem, you have to adopt the textbook's convention that surfaces are connected.)

(3) [AT] Exercise 3.48.

(4) Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the map

$$F(x,y,z) = \left(\frac{x^2-y^2}{1+z^2}, \frac{2xy}{1+z^2}, \frac{2z}{1+z^2} \right).$$

(a) Prove that F restricts to a smooth map $f\colon S^2 \to S^2$, where S^2 is the unit sphere.

(b) Let $p = (0,1,0) \in S^2$. Find bases for T_pS^2 and $T_{F(p)}S^2$, and compute the 2×2 matrix of df_p in terms of these bases.

(5) Let V be a 2-dimensional inner product space, and let B be a symmetric bilinear form on V. Let $\{\mathbf{x}_1, \mathbf{x}_2\}$ be an arbitrary basis for V, and denote the coefficients of the inner product in this basis by

$$E = \langle \mathbf{x}_1, \mathbf{x}_1 \rangle, \qquad F = \langle \mathbf{x}_1, \mathbf{x}_2 \rangle = \langle \mathbf{x}_2, \mathbf{x}_1 \rangle, \qquad G = \langle \mathbf{x}_2, \mathbf{x}_2 \rangle,$$

and the coefficients of B by

$$e = B(\mathbf{x}_1, \mathbf{x}_1), \qquad f = B(\mathbf{x}_1, \mathbf{x}_2) = B(\mathbf{x}_2, \mathbf{x}_1), \qquad g = B(\mathbf{x}_2, \mathbf{x}_2),$$

Prove that

$$\det A = \frac{eg - f^2}{EG - F^2},$$

where A denotes the symmetric endomorphism associated with B.

(6) Define $Q\colon \mathbb{R}^2 \to \mathbb{R}$ by $Q(x,y) = x^2 + y^2 + 4xy$.

(a) Prove that Q is a quadratic form, and find its associated symmetric bilinear form B and symmetric endomorphism A. (We are considering $\mathbb{R}^2$ as an inner product space with its standard inner product.)

(b) Determine the eigenvalues of A and find an orthonormal basis of eigenvectors.

(7) Suppose V is a 2-dimensional inner product space, $Q\colon V \to \mathbb{R}$ is a quadratic form, and λ_1, λ_2 are the eigenvalues of its associated endomorphism, labeled so that $\lambda_1 \le \lambda_2$. Let $S \subseteq V$ be the set of unit vectors. Prove that λ_1 and λ_2 are the minimum and maximum values of $Q|_S$.