Math 443 **Differential Geometry** **Spring 2013**

Assignment #4

Due 5/3/13 (for practice only; not to be handed in)

Reading:

- [AT] Section 4.6.
- [AT] Guided Problems 4.3, 4.4, 4.9, 4.16.

Written Problems:

(1) Let $S \subseteq \mathbb{R}^3$ be a regular surface and $p_0 \in \mathbb{R}^3$. Suppose $p \in S$ is a point where the distance $\|p_0 - p\|$ achieves a maximum among all points of S. Prove that the curvatures of S at p satisfy $K(p) \geq 1/\|p_0 - p\|^2$ and $|H(p)| \geq 1/\|p_0 - p\|$.

(2) If $S \subseteq \mathbb{R}^3$ is a compact regular surface contained in a closed ball of radius r, prove that there is a point $p \in S$ such that $K(p) \geq 1/r^2$ and $|H(p)| \geq 1/r$.

(3) Prove that there does not exist a compact regular surface $S \subseteq \mathbb{R}^3$ that has nonpositive Gaussian curvature everywhere.

(4) Let $\sigma\colon (-\pi/2, \pi/2) \to \mathbb{R}^2$ be given by $\sigma(t) = (\alpha(t), \beta(t))$, where

$$\alpha(t) = \tfrac{1}{2}\cos t,$$
$$\beta(t) = \int_0^t \sqrt{1 - \tfrac{1}{4}\sin^2 u}\, du.$$

(a) Prove that σ is a regular curve and a homeomorphism onto its image.

(b) Prove that the surface of revolution generated by σ has Gaussian curvature identically equal to 1, but that its principal curvatures are not constant.

(5) Let $S, S' \subseteq \mathbb{R}^3$ be the following surfaces:

$$S = \big\{(x,y,z) : x^2 + y^2 > 0,\ z = 0\big\},$$
$$S' = \big\{(x,y,z) : x^2 + y^2 = \tfrac{1}{3}z^2,\ z > 0\big\},$$

(so that S is the xy-plane with the origin removed, and S' is the upper half of a cone). Define a map $F\colon S \to \mathbb{R}^3$ by

$$F(x,y,0) = \frac{1}{2\sqrt{x^2+y^2}}\Big(x^2 - y^2,\ 2xy,\ \sqrt{3}(x^2+y^2)\Big).$$

Show that F is a local isometry from S to S'. Is it a global isometry?

(6) [AT] Exercise 4.57.