Math 444/445 **Geometry for Teachers** **Summer 2008**

Handout #4: Properties of the Real Numbers

UNDEFINED TERMS

Just as in geometry, some terms must remain undefined to avoid circularity. You know intuitively what these terms mean already; these descriptions are just meant to ensure that we're all thinking about the same things when we use the terms. Everything we need to know about these terms is contained in the properties listed below.

- ***Real number:*** Intuitively, a real number represents a point on the number line, or a (signed) distance left or right from the origin, or any quantity that has a finite or infinite decimal representation. Real numbers include integers, positive and negative fractions, and irrational numbers like π, e, and $\sqrt{2}$.
- ***Integer:*** An integer is a whole number (positive, negative, or zero).
- ***Zero:*** The number zero is denoted by 0.
- ***One:*** The number one is denoted by 1.
- ***Sum:*** The sum of two integers or real numbers x and y is denoted by $x + y$.
- ***Product:*** The product of two integers or real numbers x and y is denoted by xy or $x \cdot y$ or $x \times y$.
- ***Less than:*** To say that x is less than y, denoted by $x < y$, means intuitively that x is to the left of y on the number line.

DEFINITIONS

In all the definitions below, x and y represent arbitrary real numbers.

- The set of all real numbers is denoted by $\mathbb{R}$, and the set of all integers is denoted by $\mathbb{Z}$.
- To say that ***x and y are equal***, denoted by $x = y$, means that they are the same number. The expression $x \neq y$ means x is not equal to y.
- The numbers **2** through **9** are defined by $2 = 1 + 1$, $3 = 2 + 1$, etc. The decimal representations for other numbers are defined by the usual rules of decimal notation: for example, 23 is defined to be $2 \cdot 10 + 3$, etc.
- The ***additive inverse*** (or ***negative***) ***of x*** is the number $-x$ that satisfies $x + (-x) = 0$, and whose existence and uniqueness are guaranteed by property 4a below.
- The ***difference between x and y***, denoted by $x - y$, is the real number defined by $x - y = x + (-y)$.
- If $x \neq 0$, the ***multiplicative inverse of x*** is the number x^{-1} that satisfies $x \cdot x^{-1} = 1$, and whose existence and uniqueness are guaranteed by property 6a below.
- If $y \neq 0$, the ***quotient of x and y***, denoted by x/y, is the real number defined by $x/y = xy^{-1}$.
- A real number is said to be ***rational*** if it is equal to p/q for some integers p and q with $q \neq 0$. The set of all rational numbers is denoted by $\mathbb{Q}$.
- A real number is said to be ***irrational*** if it is not rational.
- The phrase ***x is less than or equal to y***, denoted by $x \leq y$, means $x < y$ or $x = y$.
- The phrase ***x is greater than y***, denoted by $x > y$, means $y < x$.
- The phrase ***x is greater than or equal to y***, denoted by $x \geq y$, means $x > y$ or $x = y$.
- A real number x is said to be ***positive*** if $x > 0$.

- A real number x is said to be ***negative*** if $x < 0$.
- A real number x is said to be ***nonnegative*** if $x \geq 0$.
- A real number x is said to be ***nonpositive*** if $x \leq 0$.
- A real number x is said to be ***nonzero*** if $x \neq 0$.
- A ***natural number*** is a positive integer. The set of all natural numbers is denoted by $\mathbb{N}$.
- An integer n is said to be ***even*** if there exists an integer k such that $n = 2k$.
- An integer n is said to be ***odd*** if there exists an integer k such that $n = 2k + 1$.
- For any real number x, the ***absolute value of $\boldsymbol{x}$***, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

- If x is a real number and n is a positive integer, the ***nth power of $\boldsymbol{x}$***, denoted by x^n, is the product of n factors of x. The ***square*** of x is the number $x^2 = x \cdot x$.
- If x is a nonnegative real number, the ***square root of $\boldsymbol{x}$***, denoted by $\sqrt{x}$, is the nonnegative real number whose square is x, and whose existence and uniqueness are guaranteed by property 10g below.
- If S is a set of real numbers, a real number b is said to be an ***upper bound for $\boldsymbol{S}$*** if $b \geq x$ for every x in S. It is said to be a ***least upper bound for $\boldsymbol{S}$*** if every other upper bound b' for S satisfies $b' \geq b$.

PROPERTIES

If our purpose were to understand the real numbers and the integers, we would choose a few of these properties as axioms, and use them to prove the others. For this course, though, you can treat all of these as if they were axioms. In all of the properties below, letters such as w, x, y, z represent arbitrary real numbers unless otherwise specified.

1. **Properties of equality**

   (a) If x is any real number, then $x = x$.
   (b) If $x = y$, then $y = x$.
   (c) If $x = y$ and $y = z$, then $x = z$.
   (d) If $x = y$, then x can be substituted for y in any formula without changing its meaning.
   (e) If $x = y$, then $-x = -y$.
   (f) If $x = y$ and x and y are nonzero, then $x^{-1} = y^{-1}$.
   (g) If $x = y$ and $z = w$, then $x + z = y + w$, $xz = yw$, and $x - z = y - w$.
   (h) If $x = y$ and $z = w$, and z and w are both nonzero, then $x/z = y/w$.

2. **Closure properties**

   (a) Every integer is a real number.
   (b) If x and y are integers, then so are $x + y$ and xy.
   (c) If x and y are real numbers, then so are $x + y$ and xy.

3. **Properties of zero and one**

   (a) 0 and 1 are integers.
   (b) $0 < 1$.

(c) $0 + x = x$.
(d) $0x = 0$.
(e) $1x = x$.
(f) If $xy = 0$, then $x = 0$ or $y = 0$.

4. **Properties of negatives**

   (a) If x is any real number, there is a unique real number $-x$ such that $x + (-x) = 0$.
   (b) If x is an integer, then so is $-x$.
   (c) $-0 = 0$.
   (d) $-(-x) = x$.
   (e) $-x = (-1)x$.
   (f) $(-x)y = -(xy) = x(-y)$.
   (g) $(-x)(-y) = xy$.
   (h) $-(x + y) = (-x) + (-y) = -x - y$.
   (i) $-(x - y) = y - x$.
   (j) $-(-x - y) = x + y$.

5. **Commutative, associative, and distributive laws**

   (a) $x + y = y + x$.
   (b) $xy = yx$.
   (c) $(x + y) + z = x + (y + z)$.
   (d) $(xy)z = x(yz)$.
   (e) $x(y + z) = xy + xz$.
   (f) $(x + y)z = xz + yz$.
   (g) $x + x = 2x$.
   (h) $x(y - z) = xy - xz$.
   (i) $(x - y)z = xz - yz$.
   (j) $(x + y)(z + w) = xz + xw + yz + yw$.
   (k) $(x + y)(z - w) = xz - xw + yz - yw$.
   (l) $(x - y)(z - w) = xz - xw - yz + yw$.

6. **Properties of inverses**

   (a) If x is any nonzero real number, there is a unique real number x^{-1} such that $x \cdot x^{-1} = 1$.
   (b) $1^{-1} = 1$.
   (c) $(x^{-1})^{-1} = x$ if x is nonzero.
   (d) $(-x)^{-1} = -(x^{-1})$ if x is nonzero.
   (e) $(xy)^{-1} = x^{-1}y^{-1}$ if x and y are nonzero.
   (f) $(x/y)^{-1} = y/x$ if x and y are nonzero.

7. **Properties of quotients**

   (a) $x/1 = x$.
   (b) $1/x = x^{-1}$ if x is nonzero.
   (c) $1/(-1) = -1$.

(d) $(x/y)(z/w) = (xz)/(yw)$ if y and w are nonzero.

(e) $(x/y)/(z/w) = (xw)/(yz)$ if y, z, and w are nonzero.

(f) $(xz)/(yz) = x/y$ if y and z are nonzero.

(g) $(-x)/y = -(x/y) = x/(-y)$ if y is nonzero.

(h) $(-x)/(-y) = x/y$ if y is nonzero.

(i) $x/y + z/w = (xw + yz)/(yw)$ if y and w are nonzero.

(j) $x/y - z/w = (xw - yz)/(yw)$ if y and w are nonzero.

8. **Properties of inequalities**

(a) (Trichotomy Law) If x and y are real numbers, then exactly one of the following three possibilities must hold: $x < y$, $x = y$, or $x > y$.

(b) If $x < y$, then $-x > -y$.

(c) If $x < y$ and x and y are both positive or both negative, then $x^{-1} > y^{-1}$.

(d) If $x < y$ and $y < z$, then $x < z$.

(e) If $x \leq y$ and $y \leq z$, then $x \leq z$.

(f) If $x \leq y$ and $y < z$, then $x < z$.

(g) If $x < y$ and $y \leq z$, then $x < z$.

(h) If $x < y$ and $z < w$, then $x + z < y + w$.

(i) If $x \leq y$ and $z < w$, then $x + z < y + w$.

(j) If $x \leq y$ and $z \leq w$, then $x + z \leq y + w$.

(k) If $x < y$ and $z > 0$, then $xz < yz$.

(l) If $x < y$ and $z < 0$, then $xz > yz$.

(m) If $x \leq y$ and $z \geq 0$, then $xz \leq yz$.

(n) If $x \leq y$ and $z \leq 0$, then $xz \geq yz$.

(o) If $x < y$ and $z < w$, and x, y, z, w are positive, then $xz < yw$.

(p) If $x < y$ and $z \leq w$, and x, y, z, w are positive, then $xz < yw$.

(q) If $x \leq y$ and $z \leq w$, and x, y, z, w are positive, then $xz \leq yw$.

(r) $xy > 0$ if and only if x and y are both positive or both negative.

(s) $xy < 0$ if and only if one is positive and the other is negative.

(t) If $x \leq y$ and $y \leq x$, then $x = y$.

9. **Properties of absolute values**

(a) If x is any real number, then $|x| \geq 0$.

(b) $|x| = 0$ if and only if $x = 0$.

(c) $|-x| = |x|$.

(d) $|x^{-1}| = 1/|x|$ if $x \neq 0$.

(e) $|xy| = |x|\,|y|$.

(f) $|x/y| = |x|/|y|$ if $y \neq 0$.

(g) (The triangle inequality for real numbers) $|x + y| \leq |x| + |y|$.

(h) If x and y are both nonnegative, then $|x| \geq |y|$ if and only if $x \geq y$.

(i) If x and y are both negative, then $|x| \geq |y|$ if and only if $x \leq y$.

10. **Properties of squares and square roots**

(a) If x is any real number, then $x^2 \geq 0$.
(b) $x^2 = 0$ if and only if $x = 0$.
(c) $(-x)^2 = x^2$.
(d) $(x^{-1})^2 = 1/x^2$.
(e) If x and y are positive, then $x < y \Rightarrow x^2 < y^2$.
(f) If x and y are negative, then $x < y \Rightarrow x^2 > y^2$.
(g) If x is any nonnegative real number, there exists a unique nonnegative real number $\sqrt{x}$ whose square is x.
(h) If $x^2 = y$, then $x = \pm\sqrt{y}$.
(i) If $x^2 = y^2$, then $x = \pm y$.

11. **Density and completeness properties**

(a) (Least Upper Bound Property) If S is any nonempty set of real numbers and S has an upper bound, then S has a unique least upper bound.
(b) (Density) If x and y are real numbers such that $x < y$, then there exists a rational number q such that $x < q < y$, and an irrational number r such that $x < r < y$.
(c) There does not exist a smallest positive real number or a smallest positive rational number.

12. **Properties of integers**

(a) If n is a positive integer, then $n \geq 1$.
(b) If m and n are integers such that $m > n$, then $m \geq n + 1$.
(c) There does not exist a largest integer.
(d) (The Archimedean Property) If M and ε are positive real numbers, there exists a positive integer n such that $n\varepsilon > M$.
(e) (The Induction Principle) If S is a set of positive integers that contains 1, and that contains $n + 1$ whenever it contains n, then S contains all the positive integers.
(f) (The Well-Ordering Principle) Every nonempty set of positive integers contains a smallest number.

13. **Even and odd integers**
In each of the following statements, m and n are assumed to be integers.

(a) Every integer is either even or odd.
(b) If m and n are both odd or both even, then $m + n$ is even.
(c) If m is odd and n is even, then $m + n$ is odd.
(d) If either m or n is even, then mn is even.
(e) If m and n are both odd, so is mn.
(f) If n is even, so is n^2.
(g) If n is odd, so is n^2.

14. **Laws of exponents**
In each of the following statements, m and n are assumed to be positive integers.

(a) $x^n y^n = (xy)^n$.
(b) $x^{m+n} = x^m x^n$.
(c) $(x^m)^n = x^{mn}$.
(d) $x^n / y^n = (x/y)^n$ if y is nonzero.