Supplement A: Rays, Angles, and Betweenness

This handout is meant to be read in place of Sections 5.6–5.7 in Venema's text [V]. You should read these pages after reading Venema's Section 5.5.

Betweenness of Points

By definition, a point B is *between* two other points A and C if all three points are collinear and $AB+BC=AC$. Although this definition is unambiguous and easy to state, it is not always easy to work with in proofs, because we may not always know what the distances AB, BC, and AC are.

There are other ways of thinking about betweenness of points using coordinate functions, which are often much more useful. Before stating them, let us establish some terminology.

We say that two or more mathematical statements are ***equivalent*** if any one of them implies all the others. For example, if P and Q are mathematical statements, then to say they are equivalent is to say that P implies Q and Q implies P, or to put it another way, "P if and only if Q." Given four statements P, Q, R, and S, if we wish to prove them all equivalent, we don't have to prove that every statement implies all of the others; for example, it would suffice to prove that $P \Rightarrow Q \Rightarrow R \Rightarrow S \Rightarrow P$ (that is, $P \Rightarrow Q$, $Q \Rightarrow R$, $R \Rightarrow S$, and $S \Rightarrow P$), for then if any one of the four statements is true, we can combine these implications to arrive at any other.

If x, y, z are three real numbers, we say that ***y is between x and z*** if either $x < y < z$ or $x > y > z$.

Theorem A.1 (Betweenness Theorem for Points). *Suppose A, B, and C are distinct points all lying on a single line ℓ. Then the following statements are equivalent:*

(*a*) $AB + BC = AC$ (*i.e.,* $A * B * C$).

(*b*) *B lies in the interior of the line segment $\overline{AC}$.*

(*c*) *B lies on the ray $\overrightarrow{AC}$ and $AB < AC$.*

(*d*) *For any coordinate function $f : \ell \to \mathbb{R}$, the coordinate $f(B)$ is between $f(A)$ and $f(C)$.*

Proof. We will prove (a) $\Leftrightarrow$ (b), (a) $\Leftrightarrow$ (c), and (a) $\Leftrightarrow$ (d).

The equivalence of (a) and (b) is just another way of restating the definition of interior points of a segment:

$$\begin{aligned} B \text{ is an interior point of } \overline{AC} &\Leftrightarrow B \in \overline{AC} \text{ and } B \neq A \text{ and } B \neq C \\ &\Leftrightarrow A * B * C \\ &\Leftrightarrow AB + BC = AC. \end{aligned}$$

Next, we will prove (a) $\Rightarrow$ (c). Assuming $A * B * C$, we conclude that $B \in \overline{AC}$ by definition of segments, and therefore $B \in \overrightarrow{AC}$ by definition of rays. The fact that $AB + BC = AC$ implies by algebra that $AB = AC - BC$, which is strictly less than AC because $BC > 0$.

Now we will prove (c) $\Rightarrow$ (a). Assuming (c), the fact that $B \in \overrightarrow{AC}$ means by definition that either $B \in \overline{AC}$ or $A * C * B$. If the latter is true, then $AC + CB = AB$. But then our assumption that $AB < AC$ implies $AC + CB < AC$, and subtracting AC from both sides we conclude that $CB < 0$, a contradiction. So the only remaining possibility is that $B \in \overline{AC}$. Since B is not equal to A or C, we must have $A * B * C$.

The next step is to prove that (a) $\Rightarrow$ (d). Suppose $AB + BC = AC$, and let $f : \ell \to \mathbb{R}$ be any coordinate function for ℓ. For convenience, write $a = f(A)$, $b = f(B)$, and $c = f(C)$, so that $AB = |b-a|$, $AC = |c-a|$, and $BC = |c-b|$ by the Ruler Postulate. Our assumption then becomes the following equation:

$$|b - a| + |c - b| = |c - a|. \tag{A.1}$$

Because A and C are distinct points, it follows that a and c are distinct real numbers. Thus there are two possibilities: Either $a < c$ or $a > c$. In the first case, $|c - a| = (c - a)$, and then from algebra and (A.1) we deduce

$$(b - a) + (c - b) = (c - a) = |c - a| = |b - a| + |c - b|. \tag{A.2}$$

Now the right-hand side of (A.2) is strictly positive, so at least one of the terms on the left-hand side must be positive, say $(b - a) > 0$. Then $(b - a) = |b - a|$ by the definition of absolute value. Subtracting this equation from (A.2), we conclude that $(c - b) = |c - b|$, which implies that $(c - b) > 0$ as well. Thus $b > a$ and $c > b$, which proves that b is between a and c. In the second case, $|c - a| = (a - c)$, and from algebra and (A.1) we deduce

$$(a - b) + (b - c) = (a - c) = |c - a| = |b - a| + |c - b|. \tag{A.3}$$

Arguing just as before, we conclude that $a > b$ and $b > c$, which again implies that b is between a and c.

Finally, we have to prove (d) $\Rightarrow$ (a). Assuming (d), let $f : \ell \to \mathbb{R}$ be a coordinate function and write $a = f(A)$, $b = f(B)$, and $c = f(C)$ as before; our assumption means that either $a < b < c$ or $a > b > c$. In the first case, algebra implies

$$AB + BC = |b - a| + |c - b| = (b - a) + (c - b) = (c - a) = |c - a| = AC,$$

and in the second case,

$$AB + BC = |b - a| + |c - b| = (a - b) + (b - c) = (a - c) = |c - a| = AC.$$

In each case, we conclude that (a) holds. □

Corollary A.2. *If A, B, and C are three distinct collinear points, then exactly one of them lies between the other two.*

Proof. Let ℓ be the line that contains A, B, and C, and let $f : \ell \to \mathbb{R}$ be a coordinate function. If $f(A) = a$, $f(B) = b$, and $f(C) = c$, then the fact that A, B, C are distinct points implies that a, b, c are distinct real numbers. It then follows from the properties of real numbers that one of these numbers is largest and one is smallest, and therefore the remaining number lies between the largest and smallest. Then the Betweenness Theorem for Points implies that the point corresponding to this number lies between the other two points. □

Often, when defining a mathematical term, there might be a choice of several equivalent definitions. For example, we might have defined "B is between A and C" to mean "all three points lie on a line ℓ, and for any coordinate function for ℓ, the coordinate of B is between those of A and C." In fact, this is exactly the definition of betweenness that Jacobs uses, as do many other high-school geometry texts. The Betweenness Theorem for Points tells us that both definitions are equivalent, in the sense that for any three points A, B, C, the truth or falsity of $A * B * C$ is the same no matter which definition we choose. The main disadvantage of the definition in terms of coordinates is that to make use of it, it is necessary to prove that betweenness does not depend on which coordinate function is used (i.e., that it is not possible to find one coordinate function that puts the coordinate of B between those of A and C, and another that puts the coordinate of A between the other two). This is a technical issue that most high-school texts don't worry about; but since we are aiming for rigor, the definition $AB + BC = AC$ is preferable because there is no ambiguity. It then follows from the Betweenness Theorem that the coordinate criterion works for *any* coordinate function.

Now suppose $\overline{AB}$ is a line segment. A point M is called a ***midpoint of $\overline{AB}$*** if M is between A and B and $AM = MB$. The requirement that M is between A and B implies that $AM + MB = AB$, and therefore simple algebra shows that $AM = (1/2)AB = MB$.

In his Proposition I.10, Euclid proved that every segment has a midpoint by showing that it can be constructed using compass and straightedge. For us, the existence and uniqueness of midpoints can be proved much more simply; the proof is an exercise in the use of coordinate functions.

Theorem A.3 (Existence and Uniqueness of Midpoints). *Every line segment has a unique midpoint.*

Proof. Exercise A.1. □

It is important to understand why we need to prove the existence of midpoints, when we did not bother to prove the existence of other geometric objects such as segments or rays. The reason is that just defining an object does not, in itself, ensure that such an object exists. For example, we might have defined "the first point in the interior of $\overline{AB}$" to be the point F in the interior of $\overline{AB}$ such that for every other point $C \neq F$ in the interior of $\overline{AB}$, we have $A * F * C$. This is a perfectly unambiguous definition, but unfortunately there is no such point, just as there is no "smallest number between 0 and 1." If we wish to talk about "the midpoint" of a segment, we have to prove that it exists.

On the other hand, if we define a *set*, as long as the definition stipulates unambiguously what it means for a point to be in that set, the set exists, even though it might be empty. Thus the definitions of *segments* and *rays* given in [V] need no further elaboration, because they yield perfectly well-defined sets. (Of course, in order for the definitions to describe anything interesting, we might wish to verify that they are nonempty. You should be able to use the Ruler Postulate to show that every line segment contains infinitely many points, as does every ray. But this is not necessary for the definition to be meaningful.)

Rays

If A and B are two distinct points, Venema has defined the ***ray*** $\overrightarrow{AB}$ to be the set of points P such that either $P \in \overline{AB}$ or $A * B * P$. If we expand out the definition of $\overline{AB}$, we see that this is equivalent to

$$P \in \overrightarrow{AB} \quad \Leftrightarrow \quad P = A \text{ or } P = B \text{ or } A * P * B \text{ or } A * B * P. \tag{A.4}$$

We define an ***interior point of*** $\overrightarrow{\boldsymbol{AB}}$ to be any point $P \in \overrightarrow{AB}$ that is not equal to A. The next theorem gives a useful alternative characterization of interior points of rays. To say that two real numbers ***have the same sign*** is to say that either both are positive or both are negative.

Theorem A.4 (Ray Theorem). *Suppose A and B are distinct points, and f is a coordinate function for the line $\overleftrightarrow{AB}$ satisfying $f(A) = 0$. Then a point $P \in \overleftrightarrow{AB}$ is an interior point of $\overrightarrow{AB}$ if and only if its coordinate has the same sign as that of B.*

Proof. Let A, B, and f be as in the hypothesis of the theorem. First assume that P is an interior point of $\overrightarrow{AB}$. Because $P \neq A$, (A.4) shows that there are three possibilities for P:

- If $P = B$, then obviously $f(P)$ and $f(B)$ have the same sign, because they are equal.
- If $A * P * B$, then the Betweenness Theorem for Rays implies that either $f(A) < f(P) < f(B)$ or $f(A) > f(P) > f(B)$. In either case, since $f(A) = 0$, we conclude that $f(P)$ and $f(B)$ have the same sign.
- If $A * B * P$, the same reasoning shows that $f(A) < f(B) < f(P)$ or $f(A) > f(B) > f(P)$, and again in both cases $f(B)$ and $f(P)$ have the same sign.

Conversely, suppose $f(P)$ and $f(B)$ have the same sign. Let's assume for starters that both coordinates are positive. By the Trichotomy Law for real numbers, there are three cases: Either $f(P) < f(B)$, $f(P) = f(B)$, or $f(P) > f(B)$. Because $f(A) = 0$, in the first case, we conclude that $f(A) < f(P) < f(B)$, so $A * P * B$ by the Betweenness Theorem. In the second case, $P = B$, and in the third, $f(P) > f(B) > f(A)$, so $P * B * A$. In all three cases, P is in the interior of $\overrightarrow{AB}$. The other possibility, that both $f(P)$ and $f(B)$ are negative, is handled similarly. □

Corollary A.5. *If A and B are distinct points, and f is a coordinate function for the line $\overleftrightarrow{AB}$ satisfying $f(A) = 0$ and $f(B) > 0$, then $\overrightarrow{AB} = \{P \in \overleftrightarrow{AB} : f(P) \geq 0\}$.*

Proof. Exercise A.2. □

Corollary A.6. *If A, B, and C are distinct collinear points, then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays if and only if $B * A * C$, and otherwise they are equal.*

Proof. Exercise A.3. ☐

Corollary A.7 (Segment Construction Theorem). *If $\overline{AB}$ is a line segment and $\overrightarrow{CD}$ is a ray, there is a unique interior point $E \in \overrightarrow{CD}$ such that $\overline{CE} \cong \overline{AB}$.*

Proof. Exercise A.4. ☐

The next theorem expresses a property of rays that seems geometrically "obvious": If a ray starts out on a line and goes to one side of the line, it cannot cross over to the other side. This will be important in our study of angles, among many other things. We call it the Y-Theorem because the drawing that goes with it (Fig. 1) is suggestive of the letter Y.

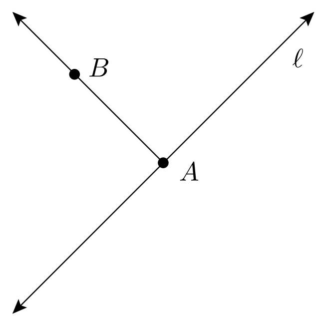

Figure 1: Setup for the Y-Theorem.

Theorem A.8 (The Y-Theorem). *Suppose ℓ is a line, A is a point on ℓ, and B is a point not on ℓ. Then every interior point of $\overrightarrow{AB}$ is on the same side of ℓ as B.*

Proof. Suppose P is an arbitrary interior point on $\overrightarrow{AB}$, and assume for the sake of contradiction that P and B are not on the same side of ℓ. There are thus two possibilities: Either P lies on ℓ, or P is on the opposite side of ℓ from B.

In the first case, we see that P and A are two distinct points on the line ℓ. By the Incidence Postulate, this implies that $\overleftrightarrow{AP} = \ell$. But since A, B, and P are collinear by the definition of a ray, and ℓ is the only line containing P and A, it follows that $B \in \ell$ also, which contradicts the hypothesis.

The other possibility is that P and B are on opposite sides of ℓ. This means that there is an interior point of $\overline{PB}$ that lies on ℓ. But since $\overline{PB}$ is contained in $\overleftrightarrow{AB}$, and A is the only point on $\ell \cap \overleftrightarrow{AB}$, the point on $\ell \cap \overline{PB}$ must be A itself. As A is not equal to P or B, it must be an interior point of $\overline{PB}$, which means that $P * A * B$. This contradicts the fact that P lies on $\overrightarrow{AB}$. ☐

Because of the Y-Theorem, we make the following definition. Suppose ℓ is a line, A is a point on ℓ, and B is a point not on ℓ. To say that ***the ray $\overrightarrow{AB}$ lies on a certain side of ℓ*** means that every interior point of $\overrightarrow{AB}$ lies on that side. The Y-Theorem tells us that if one point of a ray lies on a certain side, then the ray lies on that side. Of course, the point A itself does not lie on either side of ℓ, but when we say a ray lies on one side, we mean that its interior points do.

Recall from [V, Def. 5.5.7] that a point B is said to be in the ***interior of $\angle AOC$*** if B and C are on the same side of $\overleftrightarrow{OA}$ and B and A are on the same side of $\overleftrightarrow{OC}$. From the Y-Theorem, we can conclude that if B is in the interior of $\angle AOC$, then every interior point on the ray $\overrightarrow{OB}$ is also in the interior. In that situation, we say that ***the ray $\overrightarrow{OB}$ lies in the interior of $\angle AOC$***.

Angle Measure

Now we are ready to introduce angle measure. Recall that Venema has defined an *angle* to be the union of two nonopposite rays that share the same endpoint. Note that he did not rule out the possibility that the two rays might be *the same ray*; because the union of a ray with itself is just a ray, Venema would allow a single ray to be considered as an angle. There are no circumstances in which it is useful to work with such a "degenerate angle," so to avoid having to deal with this special case, we are going to rule it out. Thus we officially redefine an angle as follows: An ***angle*** is the union of two *distinct*, nonopposite rays that share the same endpoint.

In our approach to Euclidean geometry, *angle measure*, like *distance*, is an undefined term. Its meaning will be captured in the *Protractor Postulate* below. In order to state the postulate, we introduce the following definition. If A, O, and B are three noncollinear points, the ***half-rotation of rays determined by A, O, and B***, denoted by $\mathrm{HR}(A,O,B)$, is the set consisting of the following rays (see Fig. 2):

- The ray $\overrightarrow{OA}$,
- The ray opposite to $\overrightarrow{OA}$, and
- Every ray on the same side of $\overleftrightarrow{OA}$ as B.

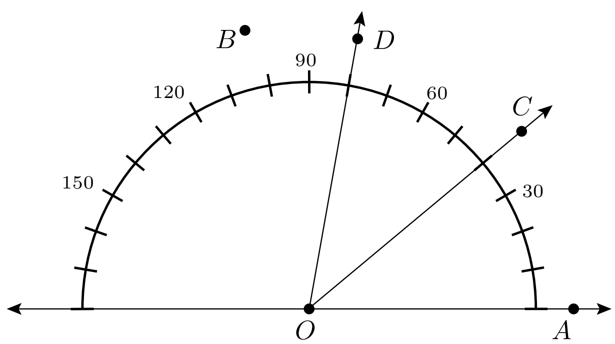

Figure 2: Rays in a half-rotation.

It is important to be clear that a half-rotation is a set of *rays*, not a set of points. (If we took the *union* of all the rays in a half-rotation, we would get a set of points, namely a half-plane together with the line $\overleftrightarrow{OA}$; but that is not what a half-rotation refers to.) It is also important to note that the set $\mathrm{HR}(A,O,B)$ depends on the order in which the points A, O, and B are listed; so in particular, $\mathrm{HR}(A,O,B)$ and $\mathrm{HR}(B,O,A)$ are *different* sets of rays. The point B is included in the definition only to stipulate which side of $\overleftrightarrow{OA}$ we are considering; any other point on the same side of that line would determine the same half-rotation.

Axiom A.9 (The Protractor Postulate). *For every angle $\angle ABC$ there exists a real number $\mu\angle ABC$, called the **measure of $\angle ABC$**. For every half-rotation $\mathrm{HR}(A,O,B)$, there is a one-to-one correspondence g from $\mathrm{HR}(A,O,B)$ to the interval $[0,180] \subset \mathbb{R}$, which sends $\overrightarrow{OA}$ to 0 and sends the ray opposite $\overrightarrow{OA}$ to 180, and such that if $\overrightarrow{OC}$ and $\overrightarrow{OD}$ are any two distinct, nonopposite rays in $\mathrm{HR}(A,O,B)$, then*

$$\mu\angle COD = \left| g(\overrightarrow{OD}) - g(\overrightarrow{OC}) \right|.$$

If m is the measure of $\angle ABC$, we write $\mu\angle ABC = m^\circ$ and say that $\angle ABC$ measures ***m degrees***. Of course, as you are aware from calculus, it is possible to measure angles using other scales such as radians; if radian measure is desired, one can just change the Protractor Postulate so that the number 180 is replaced by π. Since degrees are used exclusively in high-school geometry courses, we will stick with them.

Given any angle $\angle ABC$, there are many different half-rotations containing both of its rays; for simplicity, if we wish, we can always use the half-rotation $\mathrm{HR}(A,B,C)$ defined by the points A, B, and C themselves.

Part of what is being asserted by the Protractor Postulate is that the angle measure of $\angle ABC$ is a well-defined number, which is independent of the half-rotation used to calculate it. Intuitively, this reflects the fact that no matter where we place our 180° protractor, as long as its center is on the vertex of the angle and its scale intersects both sides of the angle, we will obtain the same value for the angle's measure. Also, since $\angle ABC$ and $\angle CBA$ represent the same angle (i.e., the same union of rays), they have the same measure. In particular, we do not distinguish between "clockwise" and "counterclockwise" angles. (In fact, it is not even clear what those words could mean in the context of our axiomatic system.)

Once we have decided on a half-rotation and its corresponding function g, the number $g(\overrightarrow{OB})$ associated with a particular ray is called the ***coordinate of the ray*** with respect to the chosen half-rotation. It is important to observe that although coordinates of rays can range all the way from 0 to 180, inclusive, it is a consequence of our conventions that angle measures are always *strictly* between 0° and 180°, as the next theorem shows.

Theorem A.10. *If $\angle ABC$ is any angle, then $0° < \mu\angle ABC < 180°$.*

Proof. Let g be the coordinate function associated with $\mathrm{HR}(A, B, C)$. Then the Protractor Postulate says that

$$\mu\angle ABC = \left| g(\overrightarrow{BC}) - g(\overrightarrow{BA}) \right|.$$

The fact that g is one-to-one means that $g(\overrightarrow{BC})$ and $g(\overrightarrow{BA})$ are different numbers, so it follows that $\mu\angle ABC > 0°$. Since the coordinates of $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are both between 0 and 180, the absolute value of their difference cannot be greater than 180, and the only way it could be equal to 180 is if one of the coordinates is 180 and the other is 0. Since the only ray whose coordinate is 0 is $\overrightarrow{BA}$, and the only one whose coordinate is 180 is its opposite ray, the only way this could happen is if $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays; but part of the definition of an angle is that they are not opposite. □

We say that an angle $\angle ABC$ is a ***right angle*** if $\mu\angle ABC = 90°$, it is an ***acute angle*** if $\mu\angle ABC < 90°$, and it is a an ***obtuse angle*** if $\mu\angle ABC > 90°$. Two angles $\angle ABC$ and $\angle DEF$ are said to be ***congruent***, written $\angle ABC \cong \angle DEF$, if $\mu\angle ABC = \mu\angle DEF$.

Some simple consequence of the Protractor Postulate are worth noting.

Theorem A.11 (Angle Construction Theorem). *Let A, O, and B be noncollinear points. For every real number m such that $0 < m < 180$, there is a unique ray $\overrightarrow{OC}$ with vertex O and lying on the same side of $\overleftrightarrow{OA}$ as B such that $\mu\angle AOC = m°$.*

Proof. Suppose m is such a real number, and let g be the coordinate function associated with $\mathrm{HR}(A, O, B)$. First we will prove existence of $\overrightarrow{OC}$. The Protractor Postulate tells us that there is a unique ray $\overrightarrow{OC} \in \mathrm{HR}(A, O, B)$ such that $g(\overrightarrow{OC}) = m$. Because $m \neq 0$, this ray is not equal to $\overrightarrow{OA}$, and because $m \neq 180$, it is not opposite to $\overrightarrow{OA}$; therefore $\angle AOC$ is an angle. The Protractor Postulate then tells us that

$$\mu\angle AOC = \left| g(\overrightarrow{OC}) - g(\overrightarrow{OA}) \right| = |m - 0| = m,$$

as desired.

To prove uniqueness, suppose $\overrightarrow{OC'}$ is any other ray on the same side of $\overleftrightarrow{OA}$ such that $\mu\angle AOC' = m°$. Then $\overrightarrow{OC'} \in \mathrm{HR}(A, O, B)$ by definition. If $m' = g(\overrightarrow{OC'})$, then the Protractor Postulate gives $m = \mu\angle AOC' = |m' - 0| = m'$. Since g is one-to-one, this implies that $\overrightarrow{OC'} = \overrightarrow{OC}$. □

Two angles are said to form a ***linear pair*** if they share a common side and their other two sides are opposite rays. Thus in Fig. 3, if we are given that $C * O * A$, then $\angle AOB$ and $\angle BOC$ form a linear pair because they share $\overrightarrow{OB}$ and the rays $\overrightarrow{OA}$ and $\overrightarrow{OC}$ are opposite. Two angles $\angle ABC$ and $\angle DEF$ are said to be ***supplementary*** if $\mu\angle ABC + \mu\angle DEF = 180°$.

Theorem A.12 (Linear Pair Theorem). *If two angles form a linear pair, they are supplementary.*

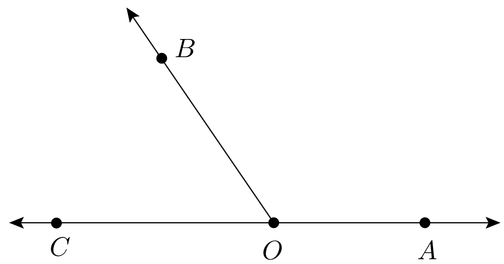

Figure 3: A linear pair.

Proof. Suppose $\angle AOB$ and $\angle BOC$ form a linear pair. Then by definition, they all lie in the half-rotation $\mathrm{HR}(A, O, B)$. Let g be the coordinate function for this half-rotation, and set $b = g(\overrightarrow{OB})$. By the Protractor Postulate, we have $g(\overrightarrow{OA}) = 0$, $g(\overrightarrow{OC}) = 180$, and so

$$\mu\angle AOB = |b - 0| = b,$$
$$\mu\angle BOC = |180 - b| = 180 - b.$$

Adding these two equations yields the result. □

Notice that the Linear Pair Theorem does *not* claim that $\mu\angle AOC = 180°$. This would make no sense, because rays $\overrightarrow{OA}$ and $\overrightarrow{OC}$ do not form an angle.

Two angles $\angle AOB$ and $\angle COD$ are said to form a pair of ***vertical angles*** if they have the same vertex O and either $\overrightarrow{OA}$ and $\overrightarrow{OC}$ are opposite rays and $\overrightarrow{OB}$ and $\overrightarrow{OD}$ are opposite rays, or $\overrightarrow{OA}$ and $\overrightarrow{OD}$ are opposite rays and $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are opposite rays. (See Fig. 4.)

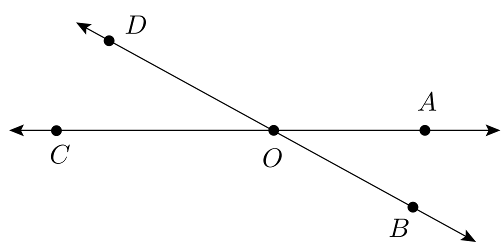

Figure 4: $\angle AOB$ and $\angle COD$ are vertical angles.

Theorem A.13 (Vertical Angles Theorem). *Vertical angles are congruent.*

Proof. Exercise A.5. □

Two lines ℓ and m are said to be ***perpendicular*** if they intersect at a point O, and one of the rays of ℓ with vertex O forms a $90°$ angle with one of the rays of m with vertex O. In this case, we write $\ell \perp m$.

Theorem A.14 (Four Right Angles Theorem). *If $\ell \perp m$, then ℓ and m form four right angles.*

Proof. Exercise A.6. □

If $\overline{AB}$ is a line segment, a ***perpendicular bisector of $\overline{AB}$*** is a line ℓ such that the midpoint of $\overline{AB}$ lies on ℓ and $\ell \perp \overleftrightarrow{AB}$.

Theorem A.15 (Existence and Uniqueness of Perpendicular Bisectors). *Every line segment has a unique perpendicular bisector.*

Proof. Exercise A.7. □

In the axiomatic approach to geometry, angle measures are left undefined, and all of their properties are deduced purely from the Protractor Postulate. In order to describe a *model* for the axioms, however, we would have to *define* angle measure and show that it satisfies the axiom. Angle measures can be defined for the Cartesian plane using inverse trigonometric functions, as you have seen in calculus. The explicit formulas and their rigorous justifications are somewhat complicated, so we will not go into them in detail. A couple of ways to define angle measures in the Cartesian model are sketched on page 330 of [V].

Betweenness of Rays

The idea of betweenness can be extended to rays. Suppose $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are three rays sharing a common endpoint, such that no two are equal and no two are opposite. We *could* choose to follow the pattern of betweenness of points, and declare that $\overrightarrow{OB}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ if $\mu\angle AOB + \mu\angle BOC = \mu\angle AOC$. However, things become somewhat simpler if we start with a different definition; we will show below that it yields the same result.

Instead, in the situation above, we say that ***$\overrightarrow{OB}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OC}$*** if $\overrightarrow{OB}$ lies in the interior of $\angle AOC$; or in other words if B (or any other interior point of $\overrightarrow{OB}$) is on the same side of $\overleftrightarrow{OA}$ as C and on the same side of $\overleftrightarrow{OC}$ as A. The idea is illustrated in Fig. 5. We symbolize this by $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$.

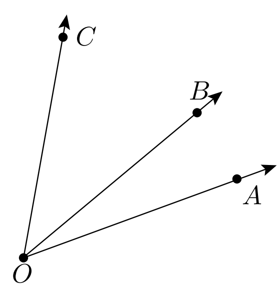

Figure 5: Betweenness of rays.

The next theorem shows that there is a close relationship between betweenness of rays and betweenness of points.

Theorem A.16 (Betweenness vs. Betweenness). *Let A, O, and C be three noncollinear points and let B be a point on the line $\overleftrightarrow{AC}$. The point B is between points A and C if and only if the ray $\overrightarrow{OB}$ is between rays $\overrightarrow{OA}$ and $\overrightarrow{OC}$.* (*See Fig. 6.*)

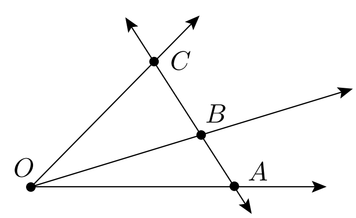

Figure 6: Betweenness vs. betweenness.

Proof. Assume first that B is between A and C. This means that $B \in \overrightarrow{AC}$, so B and C are on the same side of $\overleftrightarrow{OA}$ by the Y-theorem. The same argument shows that A and B are on the same side of $\overleftrightarrow{OC}$. This implies that $\overrightarrow{OB}$ lies in the interior of $\angle AOC$, so $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$.

Conversely, assume that $\overrightarrow{OB}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OC}$. This means that B cannot be equal to A or C, so A, B, and C are three distinct collinear points. Exactly one of them is between the other two by Corollary A.2. Our assumption means that B and A are on the same side of $\overleftrightarrow{OC}$ and B and C are on the same side of $\overleftrightarrow{OA}$. In particular, $\overline{BC}$ does not intersect $\overrightarrow{OA}$, so A cannot be between B and C; and $\overline{BA}$ does not intersect $\overrightarrow{OC}$, so C cannot be between B and A. The only remaining possibility is that B is between A and C. ☐

The next lemma is a technical one that will be used in the proof of the Betweenness Theorem for Rays below.

Lemma A.17. *Suppose O, A, B, and C are four distinct points such that the rays $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are distinct and lie on the same side of $\overleftrightarrow{OA}$. Then either $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$ or $\overrightarrow{OA} * \overrightarrow{OC} * \overrightarrow{OB}$.*

Proof. Either $\overrightarrow{OB}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ or not. If it is, we are done, so suppose it is not. Since we know that C and B lie on the same side of $\overleftrightarrow{OA}$, the fact that $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$ is false means that A and B must lie on opposite sides of $\overleftrightarrow{OC}$. This implies that $\overline{AB}$ contains an interior point E on the line $\overleftrightarrow{OC}$ (Fig. 7). By

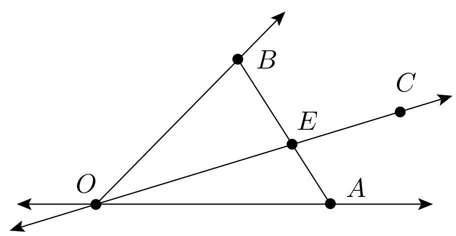

Figure 7: Proof of Lemma A.17.

the Betweenness vs. Betweenness Theorem, it follows that $\overrightarrow{OE}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. If we can show that $\overrightarrow{OE} = \overrightarrow{OC}$, then it follows that $\overrightarrow{OA} * \overrightarrow{OC} * \overrightarrow{OB}$ and the theorem is true in this case as well.

Note that B and C lie on the same side of $\overleftrightarrow{OA}$ by hypothesis, and B and E also lie on the same side by the Y-theorem. Thus C and E lie on the same side of $\overleftrightarrow{OA}$. This means that O cannot be between C and E (for if it were, C and E would lie on opposite sides of $\overleftrightarrow{OA}$), so Corollary A.6 shows that $\overrightarrow{OC} = \overrightarrow{OE}$ as claimed. ☐

Like betweenness of points, there are other equivalent characterizations of betweenness of rays. The next theorem is a central result about betweenness of rays. Its proof is a bit longer than its counterpart for points, but it will pay off by being extremely useful in all of our work in geometry. Notice the parallel between this theorem and the betweenness theorem for points.

Theorem A.18 (Betweenness Theorem for Rays). *Suppose O, A, B, and C are four distinct points such that no two of the rays $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are equal and no two are opposite. Then the following statements are equivalent:*

(*a*) $\mu\angle AOB + \mu\angle BOC = \mu\angle AOC$.

(*b*) *$\overrightarrow{OB}$ lies in the interior of $\angle AOC$ (i.e., $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$).*

(*c*) *$\overrightarrow{OB}$ lies in the half-rotation* $\mathrm{HR}(A, O, C)$ *and $\mu\angle AOB < \mu\angle AOC$.*

(*d*) *$\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ all lie in some half-rotation, and if g is the coordinate function cooresponding to any such half-rotation, the coordinate $g\big(\overrightarrow{OB}\big)$ is between $g\big(\overrightarrow{OA}\big)$ and $g\big(\overrightarrow{OC}\big)$.*

Proof. We will prove (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) $\Rightarrow$ (b).

Begin by assuming (b). This implies, in particular, that B is on the same side of $\overleftrightarrow{OA}$ as C, so $\overrightarrow{OB} \in \mathrm{HR}(A, O, C)$ by definition, and the first part of (c) is proved. Let g be the coordinate function for the half-rotation $\mathrm{HR}(A, O, C)$, and let $b = g(\overrightarrow{OB})$ and $c = g(\overrightarrow{OC})$. (Note that $g(\overrightarrow{OA}) = 0$ by the Protactor Postulate.) Because $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are distinct and neither is equal to $\overrightarrow{OA}$ or its opposite, it follows from the Protractor Postulate that b and c are distinct positive numbers less that 180, and

$$\mu\angle AOB = b, \quad \mu\angle AOC = c, \quad \mu\angle BOC = |b - c|.$$

If $b > c$, then $\mu\angle BOC = b - c < b$, while if $b < c$, then $\mu\angle BOC = c - b < c$. In either case, among the angles $\angle AOB$, $\angle BOC$, and $\angle AOC$, this shows that $\angle BOC$ is not the largest of the three.

Our assumption (b) also imples that B is on the same side of $\overleftrightarrow{OC}$ as A. Therefore, reasoning exactly as in the preceding paragraph but with A and C reversed, we conclude that $\angle AOB$ is not the largest angle either. The only remaining possibility is that $\angle AOC$ is the largest, which implies in particular that $\mu\angle AOB \le \mu\angle AOC$. If the two angles were equal, this would contradict the uniqueness part of the Angle Construction Theorem, because $\overrightarrow{OB}$ and $\overrightarrow{OC}$ lie on the same side of $\overleftrightarrow{OA}$. So finally we conclude $\mu\angle AOB < \mu\angle AOC$ as desired, so (c) has been proved.

Next assume (c). Clearly all three rays $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ lie in a single half-rotation, namely $\mathrm{HR}(A, O, B)$. If we let g_0 be the coordinate function for this half-rotation, and set $b_0 = g_0(\overrightarrow{OB})$ and $c_0 = g_0(\overrightarrow{OC})$ as before, then the hypothesis guarantees that $b_0 < c_0$, so

$$\mu\angle AOB + \mu\angle BOC = b_0 + |c_0 - b_0| = b_0 + (c_0 - b_0) = c_0 = \mu\angle AOC. \tag{A.5}$$

In particular, we have shown that (a) holds. (That is not what we are aiming for right now, but it is a fact we need to use in this part of the proof.)

Now suppose $\mathrm{HR}(E, O, F)$ is any other half-rotation containing all three rays, and let g be the corresponding coordinate function. If we put $a = g(\overrightarrow{OA})$, $b = g(\overrightarrow{OB})$, and $c = g(\overrightarrow{OC})$, then the Protractor Postulate together with (A.5) ensures that

$$|b - a| + |c - b| = |c - a|.$$

Reasoning exactly as in the proof of the Betweenness Theorem for Points, this implies that b is between a and c. Thus we have proved (d).

Now assume that (d) holds. Let $\mathrm{HR}(E, O, F)$ be some half-rotation containing the three rays, let g be the corresponding coordinate function, and define a, b, and c as in the preceding paragraph. The assumption (d) means that either $a < b < c$ or $a > b > c$. In the first case, algebra and the Protractor Postulate yield

$$\mu\angle AOB + \mu\angle BOC = |b - a| + |c - a| = (b - a) + (c - a) = (c - a) = |c - a| = \mu\angle AOC.$$

The other case is handled similarly, just as in the Proof of the Betweenness Theorem for Points. This proves (a).

Finally, assume (a). First we will show that A and C are on opposite sides of $\overleftrightarrow{OB}$. If they are not, then Lemma A.17 implies that either $\overrightarrow{OB} * \overrightarrow{OA} * \overrightarrow{OC}$ or $\overrightarrow{OB} * \overrightarrow{OC} * \overrightarrow{OA}$. In the first case, the previously proved implication (b) $\Rightarrow$(c) shows that $\mu\angle AOC < \mu\angle BOC$. In the second case, the same argument shows that $\mu\angle AOC < \mu\angle AOB$. On the other hand, (a) implies that $\mu\angle AOC$ is larger than either $\mu\angle AOB$ or $\mu\angle BOC$, which is a contradiction.

Thus we know that A and C are on opposite sides of $\overleftrightarrow{OB}$. This means that $\overline{AC}$ has an interior point E that lies on $\overleftrightarrow{OB}$. The point E cannot be equal to O, because A, O, and C are not collinear. The Betweenness vs. Betweenness Theorem tells us that $\overrightarrow{OA} * \overrightarrow{OE} * \overrightarrow{OC}$.

To complete the proof, we need to show that $\overrightarrow{OE} = \overrightarrow{OB}$. To prove this, let us assume for the sake of contradiction that these two rays are not equal; the only other possibility is that they are opposite rays (Fig. 8). Let $\alpha = \mu\angle AOB$, $\beta = \mu\angle BOC$, and $\gamma = \mu\angle AOC$, so our assumption (a) becomes

$$\alpha + \beta = \gamma. \tag{A.6}$$

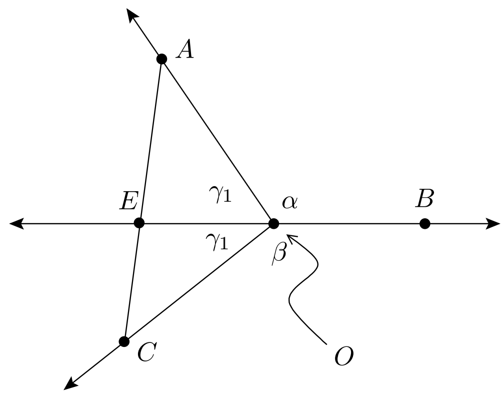

Figure 8: Proof of Lemma A.17.

Let us also put $\gamma_1 = \mu\angle AOE$ and $\gamma_2 = \mu\angle EOC$. Because $\overrightarrow{OA} * \overrightarrow{OE} * \overrightarrow{OC}$, it follows from the already proved implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) that

$$\gamma_1 + \gamma_2 = \gamma. \tag{A.7}$$

The Linear Pair Theorem tells us, on the other hand, that

$$\alpha + \gamma_1 = 180°,$$
$$\beta + \gamma_2 = 180°.$$

Adding these last two equations together and substituting (A.7), we get $\alpha + \beta + \gamma = 360°$, and then substituting (A.6) yields $\gamma + \gamma = 360°$. This implies that $\gamma = 180°$, which is a contradiction because angle measures are less than 180°. Thus $\overrightarrow{OE}$ and $\overrightarrow{OB}$ cannot be opposite rays, so we have completed the proof of (b). □

The next corollary is an analogue of Corollary A.2.

Corollary A.19. *If $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are three rays that all lie on one half-rotation and such that no two are equal and no two are opposite, then exactly one is between the other two.*

Proof. Exercise A.8. □

Note that the hypothesis that the three rays all lie in one half-rotation is essential; without it, the preceding corollary is not true. A counterexample is suggested by Fig. 9.

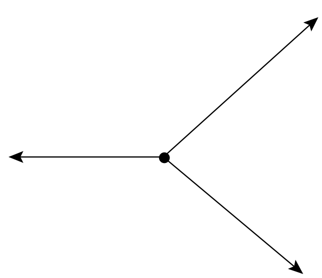

Figure 9: None of these rays is between the other two.

Suppose $\angle AOB$ is an angle. A ray $\overrightarrow{OC}$ is called an ***angle bisector of*** $\boldsymbol{\angle AOB}$ if $\overrightarrow{OA} * \overrightarrow{OC} * \overrightarrow{OB}$ and $\mu\angle AOC = \mu\angle COB$.

Theorem A.20 (Existence and Uniqueness of Angle Bisectors). *Every angle has a unique angle bisector.*

Proof. Exercise A.9. $\square$

We conclude this supplement with the *Crossbar Theorem*, another "obvious" fact that is easy to take for granted, but that is very difficult to prove without the machinery we have developed so far. It is related to Pasch's Axiom (Theorem 5.5.10 in [V]), but slightly different – whereas Pasch's Axiom says that a line that goes through one *side* of a triangle must go through one of the other sides, the Crossbar Theorem says that a ray that starts at a *vertex* and contains at least one point in the interior of that angle must go through the opposite side. Notice also the contrast between the Betweenness vs. Betweenness Theorem and the Crossbar Theorem: In the former, we *assume* that a ray through a vertex intersects the opposite side of the triangle and conclude that it is between the two corresponding sides, while in the Crossbar Theorem, we assume that the ray is between two sides of the triangle and conclude that it must intersect the opposite side.

Theorem A.21 (The Crossbar Theorem). *If $\triangle ABC$ is a triangle and $\overrightarrow{AD}$ is a ray between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then $\overrightarrow{AD}$ intersects $\overline{BC}$ (Fig. 10).*

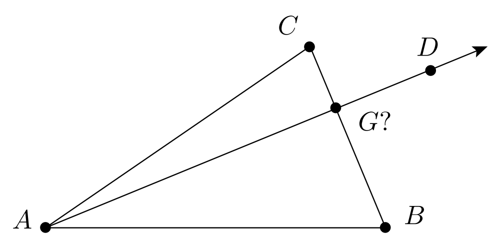

Figure 10: The Crossbar Theorem.

Proof. The fact that $\overrightarrow{AB} * \overrightarrow{AD} * \overrightarrow{AC}$ means that the following two relationships hold:

$$C \text{ and } D \text{ are on the same side of } \overleftrightarrow{AB}; \tag{A.8}$$

$$B \text{ and } D \text{ are on the same side of } \overleftrightarrow{AC}. \tag{A.9}$$

First we wish to show that B and C lie on opposite sides of $\overleftrightarrow{AD}$. Suppose not: Then

$$B \text{ and } C \text{ are on the same side of } \overleftrightarrow{AD}. \tag{A.10}$$

Combining (A.10) with (A.8) shows that $\overrightarrow{AB}*\overrightarrow{AC}*\overrightarrow{AD}$, while (A.10) together with (A.9) shows $\overrightarrow{AC}*\overrightarrow{AB}*\overrightarrow{AD}$. This is a contradiction to Corollary A.19, so our assumption is false and we conclude that B and C lie on opposite sides of $\overleftrightarrow{AD}$.

Let G be the point where $\overline{BC}$ meets $\overleftrightarrow{AD}$. Because A, B, C are noncollinear, $G \neq A$. Thus G lies either in the interior of $\overrightarrow{AD}$ or in the interior of its opposite ray. Observe that C and G are on the same side of $\overleftrightarrow{AB}$ by the Y-Theorem, and C and D are on the same side of $\overleftrightarrow{AB}$ by (A.8); thus G and D are also on the same side of $\overleftrightarrow{AB}$. It follows that A cannot be between G and D, so $\overrightarrow{AG} = \overrightarrow{AD}$ and the theorem is proved. $\square$

References

[V] Gerard A. Venema. *Foundations of Geometry.* Pearson Prentice-Hall, Upper Saddle River, NJ, 2005.

Exercises

A.1. Prove Theorem A.3 (Existence and Uniqueness of Midpoints).

A.2. Prove Corollary A.5.

A.3. Prove Corollary A.6.

A.4. Prove Corollary A.7 (the Segment Construction Theorem).

A.5. Prove Theorem A.13 (the Vertical Angles Theorem).

A.6. Prove Theorem A.14 (the Four Right Angles Theorem).

A.7. Prove Theorem A.15 (Existence and Uniqueness of Perpendicular Bisectors).

A.8. Prove Corollary A.19.

A.9. Prove Theorem A.20 (Existence and Uniqueness of Angle Bisectors).