Math 549 **Geometric Structures** **Spring 2008**

Assignment #1 (CORRECTED)

Due May 9, 2008

1. Let $E \to M$ be a k-dimensional (real or complex) vector bundle. Suppose there exists a global nonvanishing section Ω of $\Lambda^k E$, and give E the corresponding structure group $G = SL(k,\mathbb{R})$ or $SL(k,\mathbb{C})$. Prove that a connection ∇ on E is a G-connection if and only if $\nabla\Omega \equiv 0$.

2. Let $E \to M$ be a smooth vector bundle. Recall that a connection ∇ in E is said to be ***flat*** if its curvature vanishes identically. Show that ∇ is flat if and only if for each $p \in M$, there exists a parallel frame for E in a neighborhood of p.

3. Let $E \to M$ be a smooth vector bundle with structure group G, and let ∇ be a G-connection in E. Show that in a neighborhood of each $p \in M$, there is a local G-frame (s_α) for E such that $\nabla s_\alpha = 0$ at p for each α. [Hint: Start with any G-frame at p, and parallel translate along radial lines in some coordinate chart centered at p. Why is the resulting frame smooth?]

4. (a) Let G be a Lie group, and let θ be its Maurer–Cartan form. Prove that θ satisfies the following identity, known as the ***Maurer-Cartan equation***:

$$d\theta + \tfrac{1}{2}[\theta,\theta] = 0.$$

   (b) If $P = M \times G \to M$ is the canonical trivial principal G-bundle over M, show that $\pi_2^*\theta$ is a flat connection on P, where π_2 is projection onto G and θ is the Maurer–Cartan form of G.

5. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Let $P \to M$ be a smooth principal G-bundle, let ω be a connection on P, and let $H \subseteq TP$ be its horizontal distribution. For each $X \in \mathfrak{g}$, let $\widehat{X}$ denote the corresponding fundamental vector field on P.

   (a) For each $X \in \mathfrak{g}$, show that $\mathscr{L}_{\hat{X}}\omega = -\operatorname{ad}(X) \circ \omega$, where $\operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the adjoint representation of $\mathfrak{g}$, defined by $\operatorname{ad}(X)(Y) = [X,Y]$. (Recall from [ISM] that ad is the induced Lie algebra homomorphism associated with $\operatorname{Ad}\colon G \to \operatorname{GL}(\mathfrak{g})$.)

   (b) If Y is a smooth horizontal vector field on P and $X \in \mathfrak{g}$, show that $[\widehat{X}, Y]$ is horizontal.

   (c) For any $X, Y \in \mathfrak{g}$, show that $[\widehat{X}, \widehat{Y}] = \widehat{[X,Y]}$.

   (d) For any smooth k-form η on P, define the ***horizontal exterior derivative of η*** by

$$d_H\eta(X_1,\dots,X_{k+1}) = d\eta(\pi_H X_1,\dots,\pi_H X_k),$$

   where $\pi_H\colon TP \to H$ is the projection onto H with kernel V. Prove that the curvature form Ω of ω satisfies $\Omega = d_H\omega$.

6. Let M be a smooth n-manifold, and let $F(M)$ be the frame bundle of M, considered as a principal $\operatorname{GL}(n,\mathbb{R})$-bundle with projection $\pi\colon F(M) \to M$. A point $f \in F(M)$ is a basis for the tangent space T_xM, where $x = \pi(f) \in M$; such a basis can be identified with a linear isomorphism $f\colon \mathbb{R}^n \to T_xM$. Define an $\mathbb{R}^n$-valued 1-form θ on $F(M)$, called the ***soldering form***, by

$$\theta_f(X) = f^{-1}(d\pi_f(X)).$$

   (a) Show that θ is smooth.

(b) Let ω be a connection on $F(M)$, and define a smooth $\mathbb{R}^n$-valued 2-form Θ on $F(M)$ by

$$\Theta = d\theta + \omega \wedge \theta, \qquad \text{i.e.,} \qquad \Theta^i = d\theta^i + \omega^i_j \wedge \theta^j.$$

Show that

$$d_H\Theta = d\Theta + \omega \wedge \Theta = \Omega \wedge \theta,$$

where Ω is the curvature 2-form of ω.

(c) Let ∇ be the connection on TM corresponding to ω. Show that Θ is identically zero if and only if ∇ is symmetric.

7. Let M be a connected smooth manifold, let $E \to M$ be a smooth rank-k vector bundle, and let ∇ be a connection in E. Choose a basepoint $p \in M$, and for any piecewise smooth loop $\gamma\colon [0,1] \to M$ based at p, let $P_\gamma\colon E_p \to E_p$ be the linear map defined by parallel transport:

$$P_\gamma(X) = \overline{X}(1),$$

where $\overline{X}(t)$, $t \in [0,1]$, is the parallel vector field along γ satisfying $\overline{X}(0) = X$. Define a subset $H \subseteq \mathrm{GL}(E_p)$ by

$$H = \{P_\gamma : \gamma \text{ is a piecewise smooth loop based at } p\}.$$

(a) Show that H is a subgroup of $\mathrm{GL}(E_p)$, called the ***holonomy group of ∇ at p***.

(b) By choosing a basis for E_p, we may identify $\mathrm{GL}(E_p)$ with $\mathrm{GL}(k,\mathbb{R})$. Show that, up to conjugacy, the resulting subgroup $H \subseteq \mathrm{GL}(k,\mathbb{R})$ is independent of choices: If we choose any other point $p' \in M$ and any basis for $E_{p'}$, then the resulting group H' is conjugate in $\mathrm{GL}(k,\mathbb{R})$ to H.

(c) Show that E admits a reduction to H.

(d) If E admits a reduction to some subgroup $G \subseteq \mathrm{GL}(k,\mathbb{R})$ and ∇ is a G-connection, show that H is conjugate to a subgroup of G.

8. Let M, E, and ∇ be as in Problem 7. If the connection ∇ is flat, then the pair (M,∇) is called a ***flat bundle over M***.

(a) If γ is a piecewise smooth loop in M based at $p \in M$, show that $P_\gamma\colon E_p \to E_p$ depends only on the path homotopy class of γ in $\pi_1(M,p)$.

(b) Show that the map $P\colon \pi_1(M,p) \to GL(E_p)$ so defined is a representation of $\pi_1(M,p)$, called the ***holonomy representation***.

(c) We say that two flat bundles (E,∇) and $(\widetilde{E},\widetilde{\nabla})$ over M are ***equivalent over M*** if there is a bundle isomorphism $F\colon E \to \widetilde{E}$ covering the identity of M such that $F^*\widetilde{\nabla} = \nabla$, where $F^*\widetilde{\nabla}$ is the connection on E defined by $F^*\widetilde{\nabla}(\sigma) = F^{-1}(\widetilde{\nabla}(F \circ \sigma))$. For any group Γ, two representations $\rho\colon \Gamma \to GL(V)$, $\tilde{\rho}\colon \Gamma \to GL(\widetilde{V})$ are said to be ***isomorphic representations*** if there is an isomorphism $\varphi\colon V \to \widetilde{V}$ such that $\varphi \circ \rho(g) = \tilde{\rho}(g) \circ \varphi$ for all $g \in \Gamma$. Show that the holonomy representation gives a one-to-one correspondence between isomorphism classes of finite-dimensional representations of $\pi_1(M,p)$ and equivalence classes of smooth flat bundles over M.