Math 549 **Geometric Structures** **Spring 2008**

Assignment #2

Due Wednesday, June 11, 2008

The point value of each problem is shown in parentheses. For full credit, do at least 80 points worth of problems.

1. (10) Let k be a nonnegative integer and $\Omega \subseteq \mathbb{R}^n$ be an open set. Show that $H^k(\mathbb{R}^n) \cap \mathscr{E}'(\Omega) \subseteq \mathring{H}^k(\Omega)$.

2. (10) Let u be the distribution on $\mathbb{R}^n$ defined by the following locally integrable function:
$$u(x^1, \ldots, x^n) = \begin{cases} 1, & x^n > 0, \\ 0, & x^n \le 0. \end{cases}$$
Determine the distributional derivatives $\partial_i u$, $\partial_i \partial_j u$, $i, j = 1, \ldots, n$.

3. (10) Suppose $n > 2$, and let u be the distribution on $\mathbb{R}^n$ defined by the locally integrable function
$$u(x) = \frac{1}{|x|^{n-2}}.$$
Show that the following equation holds in the distribution sense:
$$\sum_{j=1}^{n} \partial_j \partial_j u = c_n \delta_0,$$
where δ_0 is the distribution defined by $(\delta_0, \varphi) = \varphi(0)$, and c_n is a constant. Determine c_n.

4. (25) For any nonnegative integer k, show that $C_c^\infty(\mathbb{R}^n)$ is dense in the Sobolev space $H^k(\mathbb{R}^n)$, as follows.
   (a) Show that compactly supported elements of $H^k(\mathbb{R}^n)$ are dense.
   (b) Choose $\rho \in C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \rho(x)\,dx = 1$, and set $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$. Show that $\int_{\mathbb{R}^n} \rho_\varepsilon(x)\,dx = 1$, and that $\rho_\varepsilon \to \delta$ as distributions (this means that $(\rho_\varepsilon, \varphi) \to (\delta, \varphi)$ for each $\varphi \in C_c^\infty(\mathbb{R}^n)$).
   (c) If $u \in H^k(\mathbb{R}^n)$ is compactly supported, define $u_\varepsilon = u * \rho_\varepsilon$. Show that $u_\varepsilon \in C_c^\infty(\mathbb{R}^n)$.
   (d) If $u \in C_c^0(\mathbb{R}^n)$, show that $u_\varepsilon \to u$ uniformly.
   (e) If $u \in L^2(\mathbb{R}^n)$, show that $u_\varepsilon \in L^2(\mathbb{R}^n)$, and
   $$\|u_\varepsilon\|_{L^2} \le \|u\|_{L^2}.$$
   [Hint: Write
   $$|u(x-y)\rho_\varepsilon(y)| = (|u(x-y)||\rho_\varepsilon(y)|^{1/2})(|\rho_\varepsilon(y)|^{1/2})$$
   and use the Cauchy-Schwartz inequality.]
   (f) Using the fact that $C_c^0(\mathbb{R}^n)$ is dense in L^2 (by standard measure theory), and interchanging limits appropriately, show that $u_\varepsilon \to u$ in L^2 norm.

(g) Show that, if $u \in H^k(\mathbb{R}^n)$ and $m \le k$,

$$\frac{\partial^m u_\varepsilon}{\partial x^{i_1} \dots \partial x^{i_m}} = \frac{\partial^m u}{\partial x^{i_1} \dots \partial x^{i_m}} * \rho_\varepsilon.$$

Use this to show that

$$\frac{\partial^m u_\varepsilon}{\partial x^{i_1} \dots \partial x^{i_m}} \to \frac{\partial^m u}{\partial x^{i_1} \dots \partial x^{i_m}}$$

in L^2 norm.

(h) Prove the result.

5. (10) Give a counterexample to the result of Problem 4 when $\mathbb{R}^n$ is replaced by the unit ball in $\mathbb{R}^n$.

6. (10) Let E be any smooth vector bundle over a compact smooth manifold M. For any nonnegative integer k, show that $\Gamma(E)$ is dense in $H^k(M, E)$.

7. (10) For any nonnegative integer k and any open set $\Omega \subseteq \mathbb{R}^n$, show that compactly supported elements of $H^k(\Omega)$ are in $\mathring{H}^k(\Omega)$.

8. (15)

   (a) For any real number s, show that a compactly supported distribution $u \in \mathscr{E}'(\mathbb{R}^n)$ is in $H^s(\mathbb{R}^n)$ if and only if it satisfies an estimate of the form

   $$(u, \varphi) \le C\|\varphi\|_{H^{-s}}, \qquad \varphi \in C_c^\infty(\mathbb{R}^n).$$

   (b) Show that every compactly supported distribution on $\mathbb{R}^n$ is in H_s for some s.

9. (10) Compute the formal adjoint of each of the following differential operators. (In each case, the metric involved is the standard Euclidean one.)

   (a) $\nabla \colon \Gamma(T^*\mathbb{R}^n) \to \Gamma(T^2\mathbb{R}^n)$.

   (b) $P \colon C^\infty(S^1) \to C^\infty(S^1)$, given by $Pu(\theta) = u'(\theta)\cos\theta$.

10. (20) This problem outlines an alternate approach to computing the symbol of the Laplace-Beltrami operator on an oriented manifold. Suppose V is an n-dimensional vector space endowed with an inner product and an orientation. Let (e_i) be any oriented orthonormal basis for V and (e^i) the dual basis. Let μ denote the volume element determined by the inner product and the orientation (thus $\mu = e^1 \wedge \dots \wedge e^n$). In terms of this basis, define a linear operator $* : \Lambda^k V \to \Lambda^{n-k} V$, the *Hodge star operator*, by setting

$$*(e^{i_1} \wedge \dots \wedge e^{i_k}) = \operatorname{sgn}(\sigma) e^{j_1} \wedge \dots \wedge e^{j_{n-k}},$$

where $(j_1, \dots, j_{n-k})$ is a the unique increasing multi-index of length $n - k$ such that

$$\{i_1, \dots, i_k, j_1, \dots, j_{n-k}\} = \{1, \dots, n\},$$

and σ is the permutation sending $(i_1, \dots, i_k, j_1, \dots, j_{n-k})$ to $(1, \dots, n)$; then extend $*$ to all of $\Lambda^k V$ by linearity.

   (a) Show that, for any k-form ω, $*\omega$ is the unique $(n-k)$-form such that $\eta \wedge *\omega = \langle \eta, \omega \rangle \mu$ for all k-forms η. This proves that $*$ is well-defined independently of choice of basis.

(b) Prove that $**\,\omega = (-1)^{k(n-k)}\omega$ for all k-forms ω.

(c) Let M be an oriented compact Riemannian manifold without boundary. Show that the L^2 inner product on $\Omega^k(M)$ can be written

$$(\omega, \eta) = \int_M \omega \wedge *\eta.$$

(d) Using Stokes's theorem, show that $d^*\eta = (-1)^{n(k+1)+1} * d * \eta$ when η is a k-form.

(e) If ξ is a 1-form and ω is a k-form, prove that

$$*(\xi \wedge \omega) = (-1)^k \xi \vee *\omega,$$

where $\xi \vee \omega = \xi^\# \lrcorner \omega$. [Hint: it might help to prove the relation

$$\xi \vee (\omega \wedge \eta) = (\xi \vee \omega) \wedge \eta + (-1)^k \omega \wedge (\xi \vee \eta),$$

and use part (a).]

(f) Prove that the symbol of the Laplace-Beltrami operator is $\sigma(\Delta)(x,\xi)\omega = -|\xi|^2\omega$.

11. (20)

(a) Show that the Laplace-Beltrami operator commutes with the Hodge star operator (see Problem 10) on k-forms.

(b) Using the Hodge star operator and the Hodge theorem, prove the *Poincaré duality theorem* for smooth manifolds: if M is a smooth, compact, oriented n-manifold, then for any $0 \le k \le n$, the map $P\colon H^k_{dR}(M) \to (H^{n-k}_{dR}(M))^*$ given by

$$P[\omega][\eta] = \int_M \omega \wedge \eta$$

is an isomorphism. Thus $\dim H^k_{dR}(M) = \dim H^{n-k}_{dR}(M)$.