Math 549 **Complex Manifolds** **Spring 2019**

ASSIGNMENT #3

DUE 5/27/19

1. If $\mathscr{S} \to M$ is a sheaf (viewed as a stalk space), $U \subseteq M$ is open, and $\sigma, \tau \in \mathscr{S}(U)$ are sections, show that the set $\{x \in U : \sigma(x) = \tau(x)\}$ is open. If U is connected and $\sigma(p) = \tau(p)$ for some $p \in M$, does this imply that $\sigma \equiv \tau$ on U?

2. Let M be a topological manifold and let $\mathscr{S}$, $\mathscr{T}$ be sheaves over M (viewed as stalk spaces). Show that every sheaf homomorphism $F\colon \mathscr{S} \to \mathscr{T}$ is a local homeomorphism.

3. Let M be a smooth manifold, and let $H^p(M; \underline{\mathbb{R}})$ denote sheaf cohomology with coefficients in the constant sheaf $\underline{\mathbb{R}} \to M$. Let $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ be an indexed open covering of M such that each nonempty finite intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ is contractible. By threading through the proof of the generalized de Rham theorem, show that the de Rham isomorphisms $\mathscr{I}_1\colon H^1_{\mathrm{dR}}(M) \to H^1(M; \underline{\mathbb{R}})$ and $\mathscr{I}_2\colon H^2_{\mathrm{dR}}(M) \to H^2(M; \underline{\mathbb{R}})$ can be described as follows.

   (a) Let η be a closed 1-form on M. For each α, there is a smooth function u_α on U_α such that $\eta|_{U_\alpha} = du_\alpha$. Then $a_{\alpha\beta} = u_\beta|_{U_\alpha \cap U_\beta} - u_\alpha|_{U_\alpha \cap U_\beta}$ defines a 1-cocycle on $\mathscr{U}$ with coefficients in $\underline{\mathbb{R}}$, and $\mathscr{I}_1[\eta] = [a]$.

   (b) Let η be a closed 2-form on M. For each α, there is a smooth 1-form φ_α on U_α such that $\eta|_{U_\alpha} = d\varphi_\alpha$; and for each α and β such that $U_\alpha \cap U_\beta \neq \varnothing$, there is a smooth function $u_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ such that $\varphi_\beta|_{U_\alpha \cap U_\beta} - \varphi_\alpha|_{U_\alpha \cap U_\beta} = du_{\alpha\beta}$. Then $a_{\alpha\beta\gamma} = \left(u_{\alpha\beta} + u_{\beta\gamma} + u_{\gamma\alpha}\right)\big|_{U_\alpha \cap U_\beta \cap U_\gamma}$ defines a 2-cocycle on $\mathscr{U}$ with coefficients in $\underline{\mathbb{R}}$, and $\mathscr{I}_2[\eta] = [a]$.

4. Let M be a complex manifold. A smooth, real-valued function u on M is said to be ***pluriharmonic*** if in any holomorphic coordinates, u is harmonic (in the usual one-complex-variable sense) as a function of each complex coordinate when the others are held fixed. Show that the following are equivalent.

   (a) u is pluriharmonic.

   (b) $\partial\overline{\partial} u = 0$.

   (c) For every holomorphic embedding $j\colon D \hookrightarrow M$ of the unit disk D into M, $j^* u$ is harmonic (in the usual sense) on D.

   (d) In a neighborhood of each point, u is the real part of a holomorphic function.

5. Let M be a complex manifold, and let $\mathscr{P}$ denote the sheaf of (germs of) pluriharmonic functions on M. For each $q \geq 1$, let $\mathscr{F}^q$ denote the sheaf of real $(q+1)$-forms whose $(q+1, 0)$ and $(0, q+1)$-parts are zero; in other words, $\mathscr{F}^q$ is the sheaf of real-valued forms in $\mathscr{E}^{q,1} \oplus \cdots \oplus \mathscr{E}^{1,q}$. Show that the following sheaf sequence is exact:

$$0 \to \mathscr{P} \hookrightarrow \mathscr{E}^0 \xrightarrow{i\partial\overline{\partial}} \mathscr{F}^1 \xrightarrow{d} \mathscr{F}^2 \xrightarrow{d} \dots \xrightarrow{d} \mathscr{F}^q \xrightarrow{d} \dots .$$

Conclude that for $q \geq 2$, $H^q(M;\mathscr{P})$ is isomorphic to the kernel of $d\colon \mathscr{F}^q(M) \to \mathscr{F}^{q+1}(M)$ modulo the image of $d\colon \mathscr{F}^{q-1}(M) \to \mathscr{F}^q(M)$. State the analogous result for $q = 1$. [Hint: For the proof of exactness at $\mathscr{F}^q$, if β is a local section of $\mathscr{F}^q$ and $\beta = d\alpha$, write $\alpha = \alpha^{(q,0)} + \widetilde{\alpha} + \alpha^{(0,q)}$ with $\widetilde{\alpha}$ a section of $\mathscr{F}^{q-1}$, and show that locally $d\alpha^{(q,0)} = d\overline{\partial}\sigma$ for some $(q-1,0)$ form σ.]

6. Let M be a complex manifold, and let E be a holomorphic vector bundle over M.

   (a) Show that the operator $\overline{\partial}\colon \Gamma(E) \to \Gamma(\Lambda^{0,1}M \otimes E)$ defined in class satisfies the following two properties.

      i. $\overline{\partial}(f\sigma) = (\overline{\partial}f) \otimes \sigma + f\overline{\partial}\sigma$.

      ii. $\overline{Z}(\overline{W}\sigma) - \overline{W}(\overline{Z}\sigma) = [\overline{Z},\overline{W}]\sigma$, where $\overline{Z}\sigma$ is shorthand for the section of E obtained by inserting $\overline{Z}$ into the $\Lambda^{0,1}$ slot of $\overline{\partial}\sigma$.

   (b) Show that $\overline{\partial}$ extends to an operator $\overline{\partial}\colon \Gamma(\Lambda^{p,q}M \otimes E) \to \Gamma(\Lambda^{p,q+1}M \otimes E)$ satisfying

$$\overline{\partial}(\alpha \otimes \sigma) = \overline{\partial}\alpha \otimes \sigma + (-1)^{p+q}\alpha \wedge \overline{\partial}\sigma,$$

   where α is a smooth (p,q)-form, σ is a smooth section of E, and the wedge product is between the differential form components of α and $\overline{\partial}\sigma$.

   (c) Show that $\overline{\partial} \circ \overline{\partial} = 0$.

7. (Optional) Let M be a complex manifold, and let $\pi\colon E \to M$ be a smooth complex vector bundle. A ***Cauchy-Riemann operator on E*** is a $\mathbb{C}$-linear map $\overline{\partial}\colon \Gamma(E) \to \Gamma(\Lambda^{0,1}M \otimes E)$ satisfying

   (a) $\overline{\partial}(f\sigma) = (\overline{\partial}f) \otimes \sigma + f\overline{\partial}\sigma$ for all smooth complex-valued functions f.

   (b) $\overline{Z}(\overline{W}\sigma) - \overline{W}(\overline{Z}\sigma) = [\overline{Z},\overline{W}]\sigma$ for all $\overline{Z},\overline{W} \in T''M$.

   (In part (ii), we define $\overline{Z}\sigma$ as in Problem 6. It follows from that problem that every holomorphic vector bundle admits a Cauchy-Riemann operator.) If E is endowed with a Cauchy-Riemann operator, show that E has a unique structure as a holomorphic vector bundle such that the holomorphic sections of E are exactly those in the kernel of $\overline{\partial}$. [Hint: If (s_k) is a smooth local frame for E over $U \subseteq M$, show that the $(0,1)$-forms θ_k^j on U defined by $\overline{\partial}s_k = \theta_k^j \otimes s_j$ satisfy $\overline{\partial}\theta_k^j + \theta_l^j \wedge \theta_k^l = 0$. Let (z^j) be local holomorphic coordinates for U and let (z^j, b^k) be the (complex-valued) coordinates on $\pi^{-1}(U) \subseteq E$ defined by the local frame (s_k), via the correspondence $(z^j, b^k) \leftrightarrow b^k s_k|_z$. Show that there is a unique integrable complex structure on the total space of E such that $\Lambda^{1,0}E$ is locally spanned by $(\pi^* dz^j, db^j + b^k \pi^* \theta_k^j)$, and apply the Newlander-Nirenberg theorem.]