Math 549 **Complex Manifolds** **Spring 2019**

Assignment #4

Due 6/12/19

1. Let Σ be a Riemann surface and let g be a Kähler metric on Σ. If z is any local holomorphic coordinate on Σ, show that the holomorphic sectional curvature of g is equal to its Gaussian curvature, and both are equal to
$$-\frac{1}{u}\frac{\partial^2}{\partial z \partial \overline{z}} \log u,$$
where $u = g_{\mathbb{C}}(\partial/\partial z, \partial/\partial \overline{z})$.

2. Let $Q \subseteq \mathbb{CP}^2$ be the quadric curve defined by the homogeneous polynomial $z^1 z^2 - (z^3)^2$. Compute the Gaussian curvature and the area of Q in the metric obtained by restricting the Fubini-Study metric to Q.

3. (a) Let $\mathrm{U}(n,1)$ be the subgroup of $\mathrm{GL}(n+1,\mathbb{C})$ leaving invariant the following hermitian bilinear form:
$$H = dz^1 \otimes d\overline{z^1} + \cdots + dz^n \otimes d\overline{z^n} - dz^{n+1} \otimes d\overline{z^{n+1}}.$$
Considering the unit ball $\mathbb{B}^{2n} \subseteq \mathbb{C}^n \subseteq \mathbb{CP}^n$ as a subset of projective space, show that $\mathrm{U}(n,1)$ acts transitively on $\mathbb{B}^{2n}$ by projective transformations.

   (b) Let g be the complex hyperbolic metric on $\mathbb{B}^{2n}$, defined by the Kähler form $\omega = -\frac{i}{2}\partial\overline{\partial}\log(1-|z|^2)$. Show that g is, up to a constant multiple, the unique $\mathrm{U}(n,1)$-invariant Riemannian metric on $\mathbb{B}^{2n}$.

   (c) Show that g has constant holomorphic sectional curvature equal to -4.

4. Let M be a complex manifold of dimension n, and let g be a Kähler metric on M with constant holomorphic sectional curvature c.

   (a) Let $v, w \in T_x M$ be a pair of orthonormal vectors. Show that the (ordinary) sectional curvature of g in the direction of the plane spanned by (v,w) is given by
$$\sec(v,w) = \tfrac{1}{4}c\left(1 + 3\langle v, Jw\rangle^2\right).$$

   (b) If $n \geq 2$, show that at each point of M, the (ordinary) sectional curvatures of g take on all values between $\frac{1}{4}c$ and c, inclusive.