

## PACIFIC NORTHWEST GEOMETRY SEMINAR

OPEN PROBLEMS, WINTER 2004

JIM BRYAN: COMPUTING THE CLOSED TOPOLOGICAL VERTEX  
VIA THE GEOMETRY OF THE CREMONA TRANSFORM

**Problem 1:** Develop a mathematical theory of the "open vertex" technology from physics that computes the Gromov-Witten invariants of any toric Calabi-Yau threefold.

**Problem 2:** Find the mathematics behind the Chern-Simons/string theory duality. In particular, find a direct connection between quantum invariants of knots and Gromov-Witten theory of Calabi-Yau threefolds.

**Problem 3:** Prove the Donaldson-Thomas/Gromov-Witten correspondence that has been conjectured by Maulik-Nekrasov-Okounkov-Pandharipande.

**Problem 4:** "Symplecticize" Donaldson-Thomas theory. i.e. find a gauge theoretic construction of the Donaldson-Thomas invariants that applies in the almost Kahler category.

**Problem 5:** Develop a gluing theory for Donaldson-Thomas invariants. That is, find the analogue in Donaldson-Thomas theory (expected to exist from the GW/DT correspondence) of Symplectic Field Theory and/or relative stable maps in Gromov-Witten theory.

JON WOLFSON: HOMOLOGY, HOLOMORPHICITY AND AREA  
MINIMIZERS IN K3 SURFACES

### Problem 1

This is a technical question.

Let  $X$  be a K3 surface. Suppose that  $g$  is a Calabi-Yau metric and  $\Sigma_1$  and  $\Sigma_2$  are  $-2$ -curves (imbedded rational curves) that intersect orthogonally at a point  $p$ . Is there a deformation of  $g$ , through Calabi-Yau metrics fixing the complex structure, so that for the perturbed metrics that intersection is non-orthogonal? The deformation space of  $g$ , fixing the complex structure, can be identified with the  $g$ -harmonic  $(1, 1)$ -forms on  $X$ . Therefore the question

can be reinterpreted as a question about the values of these forms at  $p$ .

**Problem 2**

Let  $X$  be a K3 surface and  $g$  a Calabi-Yau metric. Let  $\alpha \in H_2(X, \mathbf{Z})$  and denote the self intersection of  $\alpha$  by  $\alpha \cdot \alpha$ . In joint work with M. Micallef [M-W] (described in the talk) we consider the following problem: Given a Calabi-Yau metric  $g$  and an integral homology class  $\alpha \in H_2(X, \mathbf{Z})$  is an area minimizer of  $\alpha$  given by a sum of surfaces each calibrated by some Kähler form compatible with  $g$ ? We note that if  $\alpha \cdot \alpha \geq -2$  then every area minimizer is calibrated by a Kähler form compatible with  $g$ . Also every class  $\alpha \in H_2(X, \mathbf{Z})$  can be represented by a sum of surfaces each calibrated by some Kähler form compatible with  $g$ . However, we show that there is a class  $\alpha \in H_2(X, \mathbf{Z})$  and a Calabi-Yau metric  $g$  so that no area minimizer decomposes into a sum of surfaces each calibrated by some Kähler form compatible with  $g$ . In our construction the metric  $g$  is near the boundary of the space of Calabi-Yau metrics. That is, it is close to an orbifold metric.

**Question:** Given  $\alpha \in H_2(X, \mathbf{Z})$  with  $\alpha \cdot \alpha < -2$  describe the Calabi-Yau metrics such that an area minimizer of  $\alpha$  is given by a sum of surfaces each calibrated by some Kähler form compatible with  $g$ . In particular is this set non-empty?

REFERENCES

[M-W] Micallef, M. and Wolfson, J., Area minimizers in a K3 surface and holomorphicity, preprint.

CHARLES BOYER: EINSTEIN METRICS ON EXOTIC SPHERES

- **Conjecture:** All homotopy spheres which bound parallelizable manifolds admit Sasakian-Einstein metrics.
- Let  $K_{\min}$  denote the minimal value of the sectional curvature. Can one obtain an estimate for  $K_{\min}$ ? or better a formula in terms of the weights and degree of the BP polynomial?
- Is there a bound on the dimension of the moduli space of Sasakian-Einstein metrics on a given manifold, (or more generally for appropriately normalized Einstein metrics) on a given manifold? Find formulae that depend only on dimension. Recall that  $S^{13}$  has a moduli space of Sasakian-Einstein metrics of dimension greater than  $2.1 \times 10^{13}$ .
- How is the moduli space of Sasakian-Einstein metrics related the full moduli space of Einstein metrics?

- Does the moduli space of Sasakian-Einstein metrics have an infinite number of components? This is true for the moduli space of deformation classes of positive Sasakian structures on spheres [B,Galicki,Nakamaye] (Topology 42 (2003) 981-1002), using work of Morita and Ustilovsky on distinct contact structures. However, our proofs of Sasakian-Einstein metrics on spheres only yield a finite number of deformation classes.
- Does the dimension of a component depend on  $K_{\min}$ ?