

## Open Problems

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In general one may ask how much of Theorem 1.3 in [H99] can be generalized to dimension bigger than three, in particular dimension 4. A couple of more specific questions are given below.

Let  $M$  be a closed manifold. Suppose  $(M^n, g(t)), t \in [0, \infty)$  is a solution of Ricci flow and  $\hat{g}(\hat{t})$  is the corresponding volume-normalized solution. Define

$$\hat{i}_M(\hat{t}) = \max_{x \in M} \text{injrads}_{\hat{g}(\hat{t})}(x)$$
$$\tilde{V}_\infty = \lim_{t \rightarrow \infty} \frac{\int_M dg(t)}{t^{n/2}},$$

where  $\text{injrads}_{\hat{g}(\hat{t})}(x)$  is the injectivity radius of  $\hat{g}(\hat{t})$  at  $x$ .

(1) Suppose  $\lim_{\hat{t} \rightarrow \infty} \hat{i}_M(\hat{t}) = 0$ , show that there is a constant  $C > 0$  such that  $|\text{Rm}_{g(t)}(x)| \leq \frac{C}{t}$  for all  $x, t$  and that  $(M, \hat{g}(\hat{t}))$  collapse to a lower dimensional Euclidean space. In particular this should be true when  $\hat{g}(\hat{t})$  is a homogeneous metric and  $M$  is a Lie group. [IJL] shows that the last statement is true in dimension 4 and  $\hat{g}(\hat{t})$  is of certain diagonal form.

(2) Assume  $\tilde{V}_\infty > 0$ , show that there is an open set  $U \subset M$  and  $t_j \rightarrow \infty$  such that

(2a)  $g(t_j)/t_j \rightarrow \tilde{g}_\infty$  on  $U$ ;

(2b)  $\tilde{g}_\infty$  is an Einstein metric with negative scalar curvature;

(2c)  $\text{Vol}(U, \tilde{g}_\infty) = \tilde{V}_\infty$  (see [FIN] for more details).

## References

- [H99] R. Hamilton, *Non-singular solutions of the Ricci flow on three-manifolds*, Comm. Anal. Geom. **7** (1999) 695-729.
- [FIN] M. Feldman, T. Ilmanen and L. Ni, *Entropy and reduced distance for Ricci expanders*, math.DG/0405036.
- [IJL] J. Isenberg, M. Jackson and P. Lu, *Ricci flow on locally homogeneous closed 4-manifolds*, math.DG/0502170.