HOMOCLINIC POINTS OF ALGEBRAIC $\mathbb{Z}^d$-ACTIONS

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ABSTRACT. Let α be an action of $\mathbb{Z}^d$ by continuous automorphisms of a compact abelian group X. A point x in X is called homoclinic for α if $\alpha^{\mathbf{n}}x \to 0_X$ as $\|\mathbf{n}\| \to \infty$. We study the set $\Delta_\alpha(X)$ of homoclinic points for α, which is a subgroup of X.

If α is expansive then $\Delta_\alpha(X)$ is at most countable. Our main results are that if α is expansive, then (1) $\Delta_\alpha(x)$ is nontrivial if and only if α has positive entropy and (2) $\Delta_\alpha(X)$ is nontrivial and dense in X if and only if α has completely positive entropy. In many important cases $\Delta_\alpha(X)$ is generated by a fundamental homoclinic point which can be computed explicitly using Fourier analysis. Homoclinic points for expansive actions must decay to zero exponentially fast, and we use this to establish strong specification properties for such actions. This provides an extensive class of examples of $\mathbb{Z}^d$-actions to which Ruelle's thermodynamic formalism applies.

The paper concludes with a series of examples which highlight the crucial role of expansiveness in our main results.

1. Introduction

An *algebraic $\mathbb{Z}^d$-action* is an action of $\mathbb{Z}^d$ by (continuous) automorphisms of a compact abelian group. The dynamics of a single group automorphism have been investigated in great detail over the past several decades. More recently, the study of algebraic $\mathbb{Z}^d$-actions for $d \geq 2$ has revealed a striking interplay between these actions and commutative algebra. In §2 we summarize those parts of this interaction needed here.

The purpose of this paper is to study the homoclinic points of algebraic $\mathbb{Z}^d$-actions. Let α be an algebraic $\mathbb{Z}^d$-action on the compact abelian group X and let 0_X denote the additive identity of X. A point $x \in X$ is homoclinic for α if $\alpha^{\mathbf{n}}x \to 0_X$ as $\|\mathbf{n}\| \to \infty$. The set $\Delta_\alpha(X)$ of all homoclinic points for α is clearly a subgroup of X which we call the homoclinic group of α. In §3 we discuss some elementary properties of the homoclinic group, including countability of $\Delta_\alpha(X)$ whenever α is expansive.

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Our two main results are contained in §4. These are that if α is an expansive algebraic $\mathbb{Z}^d$-action, then (1) $\Delta_\alpha(X)$ is nontrivial if and only if α has (strictly) positive entropy, and (2) $\Delta_\alpha(X)$ is nontrivial and dense in X if and only if α has completely positive entropy. The second result is proved by first establishing in Lemma 4.5 the density of $\Delta_\alpha(X)$ for certain "principal" expansive actions by use of Fourier analysis. For these actions $\Delta_\alpha(X)$ is generated by a single fundamental homoclinic point which can be computed explicitly. This lemma is then combined with some commutative algebra to prove (2). Recent work of Kaminker and Putnam [3], [11] has suggested a general duality in the K-theory of C^*-algebras. For a principal expansive action α on X we show that $\Delta_\alpha(X)$ is isomorphic to the dual group of X, providing a class of examples to which their duality theory applies.

Ruelle has investigated expansive topological $\mathbb{Z}^d$-actions which satisfy an orbit tracing property called specification (see [12] and [13]), showing that there is a thermodynamic formalism for such actions. In §5 we show that expansive *algebraic* $\mathbb{Z}^d$-actions with completely positive entropy always satisfy very strong specification properties, thereby providing a extensive class of examples to which the thermodynamic formalism applies.

General algebraic $\mathbb{Z}^d$-actions can be built from simpler ones using a twisted skew product construction. In §6 we use the specification properties from §5 to show that a twisted skew product is measurably isomorphic to a direct product. For $d = 1$ this fact proved useful in substantially simplifying the proof that ergodic automorphisms of compact abelian groups are measurably isomorphic to Bernoulli shifts.

Finally, we describe in §7 examples of nonexpansive algebraic $\mathbb{Z}^d$-actions which show that in general there is no relationship between entropy and the size of the homoclinic group. One of these examples has completely positive entropy and trivial homoclinic group, while another has zero entropy and uncountable homoclinic group. The latter example makes crucial use of a result from Fourier analysis about the decay of the Fourier transform of a smooth measure on a hypersurface with sufficient curvature.

2. Algebraic $\mathbb{Z}^d$-actions

In this section we review the connections between algebraic $\mathbb{Z}^d$-actions and commutative algebra.

Let $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ be the ring of Laurent polynomials with integral coefficients in the commuting variables $u_1, \dots, u_d$. We write $f \in R_d$ as $f = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) u^{\mathbf{m}}$ with $u^{\mathbf{m}} = u_1^{m_1} \cdot \dots \cdot u_d^{m_d}$ and $c_f(\mathbf{m}) \in \mathbb{Z}$ for every $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$, where $c_f(\mathbf{m}) = 0$ for all but finitely many $\mathbf{m}$.

Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. The additively-written dual group $M = \widehat{X}$ is a module over the ring R_d with operation

$$f \cdot a = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) \widehat{\alpha^{\mathbf{m}}}(a) \tag{2.1}$$

for $f \in R_d$ and $a \in M$, where $\widehat{\alpha^{\mathbf{m}}}$ is the automorphism of $M = \widehat{X}$ dual to $\alpha^{\mathbf{m}}$. In particular,

$$u^{\mathbf{m}} \cdot a = \widehat{\alpha^{\mathbf{m}}}(a) \tag{2.2}$$

for $\mathbf{m} \in \mathbb{Z}^d$ and $a \in M$. The module M is Noetherian (and hence countable) whenever α is expansive (see (4.10) and Proposition 5.4 in [14]). Conversely, if M is an R_d-module, define an algebraic $\mathbb{Z}^d$-action α_M on the compact abelian group $X_M = \widehat{M}$ by declaring $\alpha_M^{\mathbf{m}}$ to be dual to multiplication by $u^{\mathbf{m}}$ on M. Note that X_M is metrizable if and only if M is countable.

A prime ideal $\mathfrak{p} \subset R_d$ is said to be *associated with* an R_d-module M if $\mathfrak{p} = \{f \in R_d : f \cdot a = 0_M\}$ for some $a \in M$, and the module M is *associated with a prime ideal* $\mathfrak{p} \subset R_d$ if $\mathfrak{p}$ is the only prime ideal associated with M. The set of (distinct) prime ideals associated with a Noetherian R_d-module M is finite.

If α is an algebraic $\mathbb{Z}^d$-action on X, then its topological entropy $h(\alpha)$ coincides with the metric entropy $h_{\lambda_X}(\alpha)$, where λ_X is the normalized Haar measure on X. We recall the following results from [14], [8], and [17, Lemma 4.5] (cf. also [15], [4], and [16]), which show that the dynamical properties of α_M are largely controlled by the prime ideals associated to M.

Lemma 2.1. *Let M be a Noetherian R_d-module with associated prime ideals $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$.*

(1) *The following conditions are equivalent.*
 (i) α_M *is expansive;*
 (ii) $\alpha_{R_d/\mathfrak{p}_j}$ *is expansive for every* $j = 1, \dots, m$*;*
 (iii) $V_{\mathbb{C}}(\mathfrak{p}_j) \cap \mathbb{S}^d = \varnothing$ *for every* $j = 1, \dots, m$*, where*

$$V_{\mathbb{C}}(\mathfrak{p}_j) = \{\, \mathbf{z} \in (\mathbb{C}^\times)^d : f(\mathbf{z}) = 0 \ \text{ for every } \ f \in \mathfrak{p}_j \,\},$$

$$\mathbb{C}^\times = \mathbb{C} \smallsetminus \{0\} \ \textit{and} \ \mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}.$$

(2) *The following conditions are equivalent.*
 (i) α_M *is mixing (with respect to Haar measure);*
 (ii) $\alpha_{R_d/\mathfrak{p}_j}$ *is mixing for every* $j = 1, \dots, m$*;*
 (iii) $\mathfrak{p}_j \cap \{u^{\mathbf{m}} - 1 : \mathbf{0} \neq \mathbf{m} \in \mathbb{Z}\} = \varnothing$ *for every* $j = 1, \dots, m$*.*

(3) *The following conditions are equivalent.*
 (i) α_M *has positive entropy (with respect to Haar measure);*
 (ii) $\alpha_{R_d/\mathfrak{p}_j}$ *has positive entropy for some* $j = 1, \dots, m$*;*
 (iii) $\mathfrak{p}_j$ *is principal and* $\alpha_{R_d/\mathfrak{p}_j}$ *is mixing for some* $j = 1, \dots, m$*.*

(4) *The following conditions are equivalent.*
 (i) α_M *has completely positive entropy (with respect to Haar measure);*
 (ii) $\alpha_{R_d/\mathfrak{p}_j}$ *has positive entropy for every* $j = 1, \dots, m$*;*
 (iii) $\mathfrak{p}_j$ *is principal and* $\alpha_{R_d/\mathfrak{p}_j}$ *is mixing for every* $j = 1, \dots, m$*.*

(5) *There exists a Noetherian R_d-module $N \supset M$ with the following properties.*
 (i) $h(\alpha_N) = h(\alpha_M)$*;*

(ii) *$N = N^{(1)} \oplus \cdots \oplus N^{(m)}$, where each of the modules $N^{(j)}$ has a finite sequence of submodules $N^{(j)} = N^{(j)}_{s_j} \supset \cdots \supset N^{(j)}_0 = \{0\}$ with $N^{(j)}_k / N^{(j)}_{k-1} \cong R_d/\mathfrak{p}_j$ for $k = 1, \dots, s_j$. In particular, α_N is expansive (or mixing) if and only if α_M is expansive (or mixing).*

In view of this lemma it is useful to have an explicit realization of $\mathbb{Z}^d$-actions of the form $\alpha_{R_d/\mathfrak{p}}$, where $\mathfrak{p} \subset R_d$ is a prime ideal. Let σ be the shift-action of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$ defined by

$$(\sigma^{\mathbf{m}} x)_{\mathbf{n}} = x_{\mathbf{m}+\mathbf{n}}\,, \tag{2.3}$$

and for $f = \sum_{\mathbf{m}\in\mathbb{Z}^d} c_f(\mathbf{m}) u^{\mathbf{m}} \in R_d$ put

$$f(\sigma) = \sum_{\mathbf{m}\in\mathbb{Z}^d} c_f(\mathbf{m}) \sigma^{\mathbf{m}} \colon \mathbb{T}^{\mathbb{Z}^d} \longrightarrow \mathbb{T}^{\mathbb{Z}^d}. \tag{2.4}$$

Identify R_d with the dual group of $\mathbb{T}^{\mathbb{Z}^d}$ by setting

$$\langle f, x\rangle = \exp\bigl[2\pi i (f(\sigma)x)_{\mathbf{0}}\bigr] \tag{2.5}$$

for $f \in R_d$ and $x \in \mathbb{T}^{\mathbb{Z}^d}$. A closed subgroup $X \subset \mathbb{T}^{\mathbb{Z}^d}$ is shift-invariant if and only if its annihilator $X^{\perp} = \mathfrak{a} \subset R_d$ is an ideal, in which case

$$X = X_{R_d/\mathfrak{a}} = \{\, x \in \mathbb{T}^{\mathbb{Z}^d} : f(\sigma)x = 0_{\mathbb{T}^{\mathbb{Z}^d}} \text{ for every } f \in \mathfrak{a} \,\} \tag{2.6}$$

and $\alpha_{R_d/\mathfrak{a}}$ is the restriction of σ to $X_{R_d/\mathfrak{a}} \subset \mathbb{T}^{\mathbb{Z}^d}$.

More generally, if α is an expansive algebraic $\mathbb{Z}^d$-action on X, then X is metrizable, $M = \widehat{X}$ is a Noetherian R_d-module, and there exist elements $a_1, \dots, a_n$ in M such that $M = R_d \cdot a_1 + \cdots + R_d \cdot a_n$. The surjective homomorphism $(f_1, \dots, f_n) \mapsto f_1 \cdot a_1 + \cdots + f_n \cdot a_n$ from $(R_d)^n$ to M dualizes to a continuous, injective group homomorphism $\psi \colon X \longrightarrow \widehat{(R_d)^n} = (\mathbb{T}^n)^{\mathbb{Z}^d}$ and allows us to regard X as a closed, shift-invariant subgroup of $(\mathbb{T}^n)^{\mathbb{Z}^d}$ and α as the restriction to X of the shift-action σ on $(\mathbb{T}^n)^{\mathbb{Z}^d}$. We write a typical point $x \in X \subset (\mathbb{T}^n)^{\mathbb{Z}^d}$ as $x = (x_{\mathbf{n}})$ with $x_{\mathbf{n}} = (x^{(1)}_{\mathbf{n}}, \dots, x^{(n)}_{\mathbf{n}}) \in \mathbb{T}^n$ for every $\mathbf{n} \in \mathbb{Z}^d$, or alternatively as $x = (x^{(1)}, \dots, x^{(n)})$ where $x^{(i)} \in \mathbb{T}^{\mathbb{Z}^d}$. Every character in

$$X^{\perp} \subset \widehat{(\mathbb{T}^n)^{\mathbb{Z}^d}} = \bigoplus_{\mathbb{Z}^d} \mathbb{Z}^n \cong (R_d)^n$$

is of the form

$$\langle h, x\rangle = \prod_{i=1}^{n} \langle h^{(i)}, x^{(i)}\rangle \tag{2.7}$$

for $x = (x^{(1)}, \dots, x^{(n)}) \in X \subset (\mathbb{T}^n)^{\mathbb{Z}^d}$, where $h = (h^{(1)}, \dots, h^{(n)}) \in (R_d)^n$ and $\langle h^{(i)}, x^{(i)}\rangle$ is defined by (2.5). The shift-invariance of X guarantees that

$$X^{\perp} = \{\, h \in (R_d)^n : \langle h, x\rangle = 1 \text{ for every } x \in X \,\}$$

is a submodule of $(R_d)^n$, and hence Noetherian. In particular there exist finitely many elements $h_j = (h_j^{(1)}, \dots, h_j^{(n)}) \in (R_d)^n$, $j = 1, \dots s$, which generate $X^\perp$ as an R_d-module, and which therefore satisfy that

$$X = \{\, x \in (\mathbb{T}^n)^{\mathbb{Z}^d} : \langle u^{\mathbf{m}} h_j, x\rangle = 1 \text{ for all } \mathbf{m} \in \mathbb{Z}^d \text{ and } j = 1, \dots, s \,\}. \tag{2.8}$$

For $t \in \mathbb{T}$ and $\mathbf{t} = (t^{(1)}, \dots, t^{(n)}) \in \mathbb{T}^n$ set

$$|t| = \min\{|t+k| : k \in \mathbb{Z}\}, \qquad |\mathbf{t}| = \max_{1\le i\le n} |t^{(i)}|. \tag{2.9}$$

For $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$ put

$$\|\mathbf{m}\| = \max_{1\le i\le d} |m_i| \tag{2.10}$$

and set

$$\mathcal{B}(r) = \{\, \mathbf{m} \in \mathbb{Z}^d : \|\mathbf{m}\| \le r \,\}. \tag{2.11}$$

The following proposition was proved in [17], and is a simple consequence of expansiveness.

Proposition 2.2. *Let $n \ge 1$, and let $X \subset (\mathbb{T}^n)^{\mathbb{Z}^d}$ be a closed, shift-invariant subgroup such that the restriction α of the shift-action σ on $(\mathbb{T}^n)^{\mathbb{Z}^d}$ to X is expansive. In the notation of* (2.9) *and* (2.11) *there exist constants $\varepsilon, \eta \in (0,1)$ and $C > 0$ with the following property: if $x \in X \subset (\mathbb{T}^n)^{\mathbb{Z}^d}$ satisfies that*

$$\max_{\mathbf{n} \in \mathbf{k}+\mathcal{B}(L)} |x_{\mathbf{n}}| < \varepsilon$$

for some $\mathbf{k} \in \mathbb{Z}^d$ and $L \ge 0$, then

$$|x_{\mathbf{k}}| < C\eta^L.$$

3. The homoclinic group

Our main object of study is the homoclinic group of an algebraic $\mathbb{Z}^d$-action.

Definition 3.1. Let α be an algebraic $\mathbb{Z}^d$-action on X. An element $x \in X$ is *α-homoclinic* (to the identity element 0_X of X), or simply *homoclinic*, if

$$\lim_{\|\mathbf{n}\|\to\infty} \alpha^{\mathbf{n}} x = 0_X.$$

The subgroup $\Delta_\alpha(X) \subset X$ of all α-homoclinic points is called the *homoclinic group* of α.

For $y \in X$ note that $\alpha^{\mathbf{n}}(y+x) - \alpha^{\mathbf{n}}(y) \to 0_X$ as $\|\mathbf{n}\| \to \infty$ if and only if $x \in \Delta_\alpha(X)$, so that the set of points in X asymptotic to y is exactly $y + \Delta_\alpha(X)$. Hence the homoclinic group of α determines the asymptotic behavior of α at all group elements.

See [1] for the role homoclinic points play in the general theory of dynamical systems.

If the homoclinic group of a $\mathbb{Z}^d$-action is finite, then clearly it must be trivial. Hence the homoclinic group is either trivial, countably infinite, or uncountable. The last case can occur (consider the full $\mathbb{Z}^d$-shift α_{R_d} on $\mathbb{T}^{\mathbb{Z}^d}$, or see Examples 7.3 and 7.5). However, for expansive actions the homoclinic group is always countable.

Lemma 3.2. *The homoclinic group of an expansive algebraic $\mathbb{Z}^d$-action is at most countable.*

Proof. Let α be an expansive algebraic $\mathbb{Z}^d$-action on X and ρ be a metric on X compatible with the topology of X. Fix an expansive constant $\delta > 0$ for α. This means that if $\rho(\alpha^{\mathbf{n}}x, \alpha^{\mathbf{n}}y) < \delta$ for all $\mathbf{n} \in \mathbb{Z}^d$, then $x = y$. For $j \geq 1$ and $y \in X$ define

$$E_j = \{\, x \in X : \rho(\alpha^{\mathbf{n}}x, 0_X) < \delta/2 \quad \text{for all } \mathbf{n} \text{ with } \|\mathbf{n}\| > j \,\}$$

and

$$B_j(y) = \{\, x \in X : \rho(\alpha^{\mathbf{n}}x, \alpha^{\mathbf{n}}y) < \delta/2 \quad \text{for all } \mathbf{n} \text{ with } \|\mathbf{n}\| \leq j \,\}.$$

We claim that that $|E_j \cap B_j(y)| \leq 1$ for all $y \in X$. For if x and x' are in $E_j \cap B_j(y)$, then for $\|\mathbf{n}\| > j$ we have that

$$\rho(\alpha^{\mathbf{n}}x, \alpha^{\mathbf{n}}x') \leq \rho(\alpha^{\mathbf{n}}x, 0_X) + \rho(0_X, \alpha^{\mathbf{n}}x') < \delta,$$

while for $\|\mathbf{n}\| \leq j$ we have that

$$\rho(\alpha^{\mathbf{n}}x, \alpha^{\mathbf{n}}x') \leq \rho(\alpha^{\mathbf{n}}x, \alpha^{\mathbf{n}}y) + \rho(\alpha^{\mathbf{n}}y, \alpha^{\mathbf{n}}x') < \delta.$$

Hence $x = x'$ by expansiveness. For fixed j the open cover $\{B_j(y) : y \in X\}$ of X has a finite subcover, so that E_j is finite for every $j \geq 1$. Hence $\Delta_\alpha(X) \subset \bigcup_{j=1}^{\infty} E_j$ is at most countable. □

As noted above, some condition is needed to ensure countability of the homoclinic group. In Example 7.5 we describe a nonexpansive $\mathbb{Z}^3$-action with zero entropy and uncountable homoclinic group.

For a hyperbolic toral automorphism there is a direct geometric description of its homoclinic group.

Example 3.3. (*The homoclinic group of a hyperbolic toral automorphism.*) This example shows that for the $\mathbb{Z}$-action generated by a single hyperbolic automorphism ϕ on $\mathbb{T}^k$, the homoclinic group $\Delta_\phi(\mathbb{T}^k)$ is the intersection of the stable and the unstable subgroups of ϕ, that it is dense in $\mathbb{T}^k$, and that the restriction of ϕ to $\Delta_\phi(\mathbb{T}^k)$ is isomorphic to the transpose of the dual automorphism $\widehat{\phi}$ on the dual group $\widehat{\mathbb{T}^k} \cong \mathbb{Z}^k$.

Let $\pi \colon \mathbb{R}^k \longrightarrow \mathbb{T}^k$ be the usual quotient map and $A \in GL(k, \mathbb{Z})$ be the linear hyperbolic map such that $\pi \circ A = \phi \circ \pi$. Then $\mathbb{R}^k = \mathcal{C} \oplus \mathcal{E}$, where A contracts on the subspace $\mathcal{C}$ and expands on the subspace $\mathcal{E}$. Observe that $\mathcal{C} \cap \mathbb{Z}^k = \{\mathbf{0}\}$ since there are no contracting automorphisms of nontrivial discrete groups. Similarly, $\mathcal{E} \cap \mathbb{Z}^k = \{\mathbf{0}\}$. Hence π is injective on $\mathcal{C}$, and we claim that its image $C = \pi(\mathcal{C})$ is dense in $\mathbb{T}^k$. To verify this assertion, note that C is connected, so that $\overline{C}$ is connected and compact in $\mathbb{T}^k$, hence a subtorus that is obviously invariant under ϕ. If $\overline{C} \neq \mathbb{T}^k$, then the quotient automorphism of ϕ on $\mathbb{T}^k/\overline{C}$ would be a toral automorphism all of whose eigenvalues are greater than one in modulus, which would violate preservation of Haar measure. Hence $\overline{C} = \mathbb{T}^k$, as claimed.

We next show that $\Delta_\phi = \Delta_\phi(\mathbb{T}^k) = C \cap E$. For suppose that $x \in \Delta_\phi$, and let $\mathbf{x} \in \mathbb{R}^k$ be a lift of x, so that $\pi\mathbf{x} = x$. Since π is a local homeomorphism, it follows from $\phi^j(x) \to 0$ as $j \to +\infty$ that $A^j\mathbf{x}$ is asymptotic to a lattice point $\mathbf{m} \in \mathbb{Z}^k$ in the sense that $A^j(\mathbf{x}-\mathbf{m}) \to 0$ as $j \to +\infty$. Thus $\mathbf{x} \in \mathcal{C}+\mathbf{m}$. Similarly, $\phi^j(x) \to 0$ as $j \to -\infty$ shows there is a $\mathbf{n} \in \mathbb{Z}^k$ such that $\mathbf{x} \in \mathcal{E}+\mathbf{n}$. Observe that $(\mathcal{C}+\mathbf{m}) \cap (\mathcal{E}+\mathbf{n})$ is the singleton $\{\mathbf{x}\}$ and that $x = \pi\mathbf{x} \in C \cap E$. Conversely, for every $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^k$, the point in $(\mathcal{C}+\mathbf{m}) \cap (\mathcal{E}+\mathbf{n})$ clearly projects under π to an element of Δ_ϕ. Hence $\Delta_\phi = C \cap E$.

Define $\theta\colon \mathbb{Z}^k \longrightarrow \Delta_\phi$ by $\theta(\mathbf{n}) = \pi[\mathcal{C} \cap (\mathcal{E}+\mathbf{n})]$. In the previous discussion the lift $\mathbf{x}$ could be adjusted by a lattice point so that $\mathbf{m} = \mathbf{0}$, showing that θ is surjective. It is injective since $\mathcal{E} \cap \mathbb{Z}^k = \{\mathbf{0}\}$ and π is injective on $\mathcal{C}$. Clearly $\theta \circ A = \phi \circ \theta$, so that the action of ϕ on Δ_ϕ is isomorphic to the action of A on $\mathbb{Z}^k$. The matrix of the dual automorphism $\widehat{\phi}$ on $\widehat{\mathbb{T}^k} \cong \mathbb{Z}^k$ with respect to the standard basis on $\mathbb{Z}^k$ is the transpose $A^\top$ of A, verifying the last claim of the first paragraph.

Finally, we prove that Δ_ϕ is dense in $\mathbb{T}^k$. From the above we know that $\Delta_\phi = \pi[\mathcal{C} \cap (\mathcal{E}+\mathbb{Z}^k)]$, and we claim that $\Gamma = \mathcal{C} \cap (\mathcal{E}+\mathbb{Z}^k)$ is dense in $\mathcal{C}$. First note that $\overline{\Gamma} \subset \mathcal{C}$ is A-invariant, and thus connected since otherwise A would induce a contracting automorphism of the discrete group $\overline{\Gamma}$ modulo its connected component of the identity. Hence $\overline{\Gamma}$ is a subspace of $\mathcal{C}$, and $\mathbb{Z}^k \subset \overline{\Gamma} \oplus \mathcal{E}$, thus $\overline{\Gamma} \oplus \mathcal{E} = \mathbb{R}^k$, and so $\overline{\Gamma} = \mathcal{C}$. Then density of $C = \pi(\mathcal{C})$ in $\mathbb{T}^k$ proves that $\Delta_\phi = \pi(\Gamma)$ is also dense.

We conclude this example with some remarks whose relevance will become clearer in Example 4.7. If A is the companion matrix of a monic polynomial with constant term ± 1, then it is easy to construct an explicit conjugacy in $GL(k,\mathbb{Z})$ between A and $A^\top$. Hence in this case $\mathbb{T}^k$ contains the countable group Δ_ϕ such that the action of ϕ on Δ_ϕ is isomorphic to the action of the dual automorphism $\widehat{\phi}$ on $\widehat{\mathbb{T}^k}$. However, this fails for general $A \in GL(k,\mathbb{Z})$. For example, the matrix $A = \left[\begin{smallmatrix}19 & 5\\ 4 & 1\end{smallmatrix}\right]$ is not conjugate to $A^\top$ in $GL(2,\mathbb{Z})$ (see [10, p. 81]), and so here (Δ_ϕ, ϕ) is not isomorphic to $(\widehat{\mathbb{T}^k}, \widehat{\phi})$.

Example 3.4. (*An ergodic nonhyperbolic toral automorphism having trivial homoclinic group.*) Let $A \in GL(k,\mathbb{Z})$ have characteristic polynomial $\chi_A(t)$ that is irreducible over $\mathbb{Q}$ and which has some but not all of its eigenvalues on the unit circle. There is an A-invariant splitting $\mathbb{R}^k = \mathcal{C} \oplus \mathcal{N} \oplus \mathcal{E}$, where A contracts on $\mathcal{C}$, expands on $\mathcal{E}$, and is an isometry on $\mathcal{N}$. Let ϕ be the automorphism of $\mathbb{T}^k$ induced by A and $\pi\colon \mathbb{R}^k \longrightarrow \mathbb{T}^k$ the quotient map. As in the previous example, we obtain that $\Delta_\phi(\mathbb{T}^k) = \pi(\mathcal{C}) \cap \pi(\mathcal{E})$. Now $(\mathcal{C} \oplus \mathcal{E}) \cap \mathbb{Z}^d = \{\mathbf{0}\}$, since otherwise $\chi_A(t)$ would have a proper factor with integer coefficients, contradicting its irreducibility. Hence $\Delta_\phi(\mathbb{T}^k) = \{\mathbf{0}\}$.

This is an example of a (nonexpansive) $\mathbb{Z}$-action with completely positive entropy having trivial homoclinic group.

In Example 3.3 the homoclinic group is dense. We show in the next section that for an expansive action on a nontrivial group this occurs exactly when the action has completely positive entropy. For now, let us point out one direct consequence of density.

Proposition 3.5. *Let α be an algebraic $\mathbb{Z}^d$-action on X, and suppose that $\Delta_\alpha(X)$ is dense in X. Then $\alpha^{\mathbf{n}}$ is ergodic with respect to Haar measure for every $\mathbf{n} \neq \mathbf{0}$.*

Proof. Denote $\alpha^{\mathbf{n}}$ by ϕ, and observe that since $\mathbf{n} \neq \mathbf{0}$ we have $\Delta_\alpha(X) \subset \Delta_\phi(X)$, so $\Delta_\phi(X)$ is dense in X as well. Let λ denote Haar measure on X. If ϕ were not ergodic, there would exist a nonconstant $f \in L^2(X, \lambda)$ with $f \circ \phi = f$ (λ-a.e.). Since $\Delta_\phi(X)$ is dense, there is a $t \in \Delta_\phi(X)$ and an $\epsilon > 0$ such that $E = \{x \in X : |f(x) - f_t(x)| > \epsilon\}$ has $\lambda(E) > 0$, where $f_t(x) = f(x + t)$. On the other hand,

$$(f - f_t) \circ \phi^n = f \circ \phi^n - (f \circ \phi^n)_{\phi^{-n}t} = f - f_{\phi^{-n}t},$$

and $f_{\phi^{-n}t} \to f$ in measure since $\phi^{-n}t \to 0_X$. Since ϕ preserves Haar measure, we would then obtain that

$$\begin{aligned} 0 < \lambda(E) = \lambda(\phi^{-n}E) &= \lambda(\{|(f - f_t) \circ \phi^n| > \epsilon\}) \\ &= \lambda(\{|f - f_{\phi^{-n}t}| > \epsilon\}) \to 0 \end{aligned}$$

as $n \to \infty$. This contradiction shows that ϕ is ergodic. □

The density of the homoclinic group in this proof plays a role remarkably similar to that of minimality of the horocycle flow in the proof of ergodicity of the geodesic flow. In particular, if T_t denotes the translation operator $T_t f = f_t$ and $U_\phi f = f \circ \phi$, then the commutation relation $U_\phi T_t = T_{\phi^{-1}t} U_\phi$ is crucial to the proof. This relation is analogous to the Weyl commutation relation between the horocycle and geodesic flows (see [9] for details).

We conclude this section by briefly discussing the functorial properties of Δ. Consider the category whose objects are pairs (X, α), where α is an algebraic $\mathbb{Z}^d$-action on the compact abelian group X, and whose morphisms $\theta\colon (X, \alpha) \longrightarrow (Y, \beta)$ are group homomorphisms $\theta\colon X \longrightarrow Y$ so that $\theta \circ \alpha^{\mathbf{n}} = \beta^{\mathbf{n}} \circ \theta$ for all $\mathbf{n} \in \mathbb{Z}$. For the rest of this section we write $\Delta(X, \alpha)$ instead of $\Delta_\alpha(X)$ to emphasize the functorial nature of Δ. If $\theta\colon (X, \alpha) \longrightarrow (Y, \beta)$ is a morphism, then the restriction $\Delta(\theta)$ of θ to $\Delta(X, \alpha)$ has range contained in $\Delta(Y, \beta)$. Hence Δ is a covariant functor from the category of algebraic $\mathbb{Z}^d$-actions to the category of abelian groups.

Suppose that

$$0 \longrightarrow (X, \alpha) \xrightarrow{\ \theta\ } (Y, \beta) \xrightarrow{\ \phi\ } (Z, \gamma) \longrightarrow 0$$

is a short exact sequence. It is easy to see that

$$0 \longrightarrow \Delta(X, \alpha) \xrightarrow{\Delta(\theta)} \Delta(Y, \beta) \xrightarrow{\Delta(\phi)} \Delta(Z, \gamma)$$

is also exact. However, the following example shows that exactness can fail at (Z, γ).

Example 3.6. (*A surjective morphism of algebraic $\mathbb{Z}$-actions that is not surjective on the corresponding homoclinic groups.*) Let $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and α be the $\mathbb{Z}$-action on $X = \mathbb{T}^2$ generated by the automorphism of X induced by A. Consider the morphism $\theta\colon (X, \alpha) \longrightarrow (X, \alpha)$ defined by $\theta(x_1, x_2) = (2x_1, 2x_2)$, which is clearly surjective. According to Example 3.3, $\Delta(X, \alpha)$ is a free subgroup of X that is isomorphic to $\mathbb{Z}^2$. Hence $\Delta(\theta)(\Delta(X, \alpha))$ has index 4 in $\Delta(X, \alpha)$, so that $\Delta(\theta)\colon \Delta(X, \alpha) \longrightarrow \Delta(X, \alpha)$ is not surjective.

To see the underlying reason making this example work, let $K = \ker \theta$ and y be a point in $\Delta(X, \alpha)$ that is not in the image of $\Delta(\theta)$. Then the coset $y + K$ is "homoclinic" in the sense that $\alpha^n(y + K)$ converges to $0 + K$ in the Hausdorff metric as $|n| \to \infty$. Since K is finite, there are k_1, $k_2 \in K$ such that $\alpha^n(y + k_1) \to 0_X$ as $n \to -\infty$ and $\alpha^n(y + k_2) \to 0_X$ as $n \to +\infty$. However $k_1 \neq k_2$, and so no element of $y + K$ is itself α-homoclinic.

The category of algebraic $\mathbb{Z}^d$-actions is closed under the operations of taking arbitrary direct products and inverse limits. It is routine to show that Δ commutes with both of these operations. We use this observation to show that right exactness of Δ can fail more dramatically than indicated by Example 3.6.

Example 3.7. (*A surjective morphism from an algebraic $\mathbb{Z}$-action with trivial homoclinic group onto one with dense homoclinic group.*) We use the inverse limit of a collection of $\mathbb{Z}$-actions (X_j, α_j) indexed by $j \in \mathbb{Z}$. For each $j \in \mathbb{Z}$ let $X_j = \mathbb{T}^2$, let α_j be the automorphism of X_j described in Example 3.6, and let $\theta_j\colon X_j \longrightarrow X_{j-1}$ be defined by $\theta_j(x) = 2x$ as in that example. Put

$$(X, \alpha) = \varprojlim \left(\{(X_j, \alpha_j)\}, \{\theta_j\}\right).$$

Since Δ commutes with inverse limits, we see that

$$\Delta(X, \alpha) = \varprojlim \left(\{\Delta(X_j, \alpha_j)\}, \{\Delta(\theta_j)\}\right).$$

Now each $\Delta(X_j, \alpha_j)$ is isomorphic to $\mathbb{Z}^2$, and $\Delta(\theta_j)$ is multiplication by 2. Since no element of $\mathbb{Z}^2$ except 0 is infinitely divisible by 2, it follows that $\Delta(X, \alpha)$ is trivial.

Define $\psi\colon X \longrightarrow X_0$ by $\psi\big((x_j)\big) = x_0$. Then ψ is a surjective morphism from (X, α) to (X_0, α_0), and $\Delta(X_0, \alpha_0)$ is dense in X_0 by Example 3.3.

4. Homoclinic groups of expansive algebraic $\mathbb{Z}^d$-actions

In this section we determine which expansive algebraic $\mathbb{Z}^d$-actions (where $d \geq 1$ as usual) have nontrivial homoclinic group and which have dense homoclinic group.

Theorem 4.1. *Let α be an expansive $\mathbb{Z}^d$-action by automorphisms of a compact abelian group X. Then $\Delta_\alpha(X) \neq \{0_X\}$ if and only if α has positive entropy.*

Theorem 4.2. *Let α be an expansive $\mathbb{Z}^d$-action by automorphisms of a nontrivial compact abelian group X. Then $\Delta_\alpha(X)$ is dense in X if and only if α has completely positive entropy.*

Note that Example 3.4 shows that both of these results are false if the expansiveness assumption is dropped.

For the proof of Theorem 4.1, by §2 we may assume that X is a closed, shift-invariant subgroup of $(\mathbb{T}^n)^{\mathbb{Z}^d}$ for some $n \geq 1$, and that α is the shift-action of $\mathbb{Z}^d$ on X. For every $x = (x^{(1)}, \dots, x^{(n)}) \in X \subset (\mathbb{T}^n)^{\mathbb{Z}^d}$ and $\mathbf{n} \in \mathbb{Z}^d$ we set $x_{\mathbf{n}} = (x_{\mathbf{n}}^{(1)}, \dots, x_{\mathbf{n}}^{(n)}) \in \mathbb{T}^n$ and define $|x_{\mathbf{n}}|$ and $\|\mathbf{n}\|$ by (2.9) and (2.10). The following lemma is an immediate consequence of Proposition 2.2.

Lemma 4.3. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on X and assume that $X \subset (\mathbb{T}^n)^{\mathbb{Z}^d}$ as above. There is a positive constant $\eta < 1$ such that for every $x \in \Delta_\alpha(X)$ there is a $C > 0$ such that $|x_{\mathbf{n}}| < C\,\eta^{\|\mathbf{n}\|}$ for all $\mathbf{n} \in \mathbb{Z}^d$.*

Thus homoclinic points for expansive algebraic $\mathbb{Z}^d$-actions must decay exponentially fast, and this is crucial to what follows. Note that the expansive hypothesis is necessary, since, for example, the shift-action on $\mathbb{T}^{\mathbb{Z}^d}$ has homoclinic points which decay arbitrarily slowly.

Lemma 4.4. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on X. If $\Delta_\alpha(X) \neq \{0_X\}$ then $h(\alpha) > 0$.*

Proof. We use the notations above and assume that $X \subset (\mathbb{T}^n)^{\mathbb{Z}^d}$. Since α is expansive there exists a $\delta > 0$ such that if $x = (x_{\mathbf{n}}) \in X \subset (\mathbb{T}^n)^{\mathbb{Z}^d}$ and $\sup_{\mathbf{n} \in \mathbb{Z}^d} |x_{\mathbf{n}}| < \delta$, then $x = 0_X$. If $\varepsilon, r > 0$ we say that a set $E \subset X$ is (r, ε)-*separated* if there exists, for every pair x, x' of distinct points in E, a coordinate $\mathbf{n} \in \mathcal{B}(r)$ with $|x_{\mathbf{n}} - x'_{\mathbf{n}}| \geq \varepsilon$. We denote by $s(r, \varepsilon)$ the maximum of the cardinalities of all (r, ε)-separated sets in X and observe that

$$h(\alpha) = \limsup_{r \to \infty} \frac{1}{(2r+1)^d} \log s(r, \varepsilon)$$

for every $\varepsilon \in (0, \delta)$ (see [8, Appendix A] or [15]).

If there exists a nonzero α-homoclinic point $x \in X$ we can choose an $\varepsilon \in (0, \delta)$ with $2\varepsilon < |x_{\mathbf{n}}|$ for some $\mathbf{n} \in \mathbb{Z}^d$. Adjusting by an iterate of α if necessary, we may assume that $\mathbf{n} = \mathbf{0}$. Apply Lemma 4.3 to find an $r \geq 0$ with $|x_{\mathbf{n}}| < \varepsilon$ for every $\mathbf{n} \in \mathbb{Z}^d \smallsetminus \mathcal{B}(r)$. Lemma 4.3 also allows us to find an integer $k \geq 2r$ for which

$$\sum_{\mathbf{0} \neq \mathbf{n} \in \mathbb{Z}^d} |x_{k\mathbf{n}}| < \varepsilon.$$

Let $L \geq 1$ and put, for every $\omega = (\omega_{\mathbf{n}}) \in \{0,1\}^{\mathcal{B}(L)}$,

$$y(\omega) = \sum_{\mathbf{n} \in \mathcal{B}(L)} \omega_{\mathbf{n}} \sigma^{k\mathbf{n}}(x).$$

Our choices of ε, r and k imply that the set

$$E(L) = \{y(\omega) : \omega \in \{0,1\}^{\mathcal{B}(L)}\} \subset X$$

is (kL,ε)-separated. Since $E(L)$ has cardinality $|E(L)| = 2^{(2L+1)^d}$ we obtain that

$$\begin{aligned} h(\alpha) &= \limsup_{r\to\infty} \frac{1}{(2r+1)^d} \log s(r,\varepsilon) \\ &\geq \lim_{L\to\infty} \log 2\, \frac{(2L+1)^d}{(2kL+1)^d} = \frac{\log 2}{k^d} > 0. \end{aligned}$$ □

Lemma 4.5. *Let $f \in R_d$ be a (possibly reducible) Laurent polynomial such that the $\mathbb{Z}^d$-action $\alpha = \alpha_{R_d/fR_d}$ on $X = X_{R_d/fR_d}$ is expansive and mixing, (and so has positive entropy by Lemma 2.1). Then the homoclinic group $\Delta_\alpha(X)$ is countable and dense in X.*

Furthermore there exists a group isomorphism $\tau\colon R_d/fR_d \longrightarrow \Delta_\alpha(X)$ with $\alpha^{\mathbf{n}} \cdot \tau(h) = \tau(u^{\mathbf{n}}h)$ for every $h \in R_d/fR_d$ and $\mathbf{n} \in \mathbb{Z}^d$. Therefore the $\mathbb{Z}^d$-action obtained by restricting α to $\Delta_\alpha(X)$ is isomorphic to the $\mathbb{Z}^d$-action on $\widehat{X}$ dual to α.

Proof. As in §2 we represent X as a closed, shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}^d}$ and α as the shift-action of $\mathbb{Z}^d$ to X. Let $\ell^2(\mathbb{Z}^d,\mathbb{R})$ denote the Hilbert space of square-summable real-valued functions on $\mathbb{Z}^d$, and define the convolution of $v, w \in \ell^2(\mathbb{Z}^d,\mathbb{R})$ by $(v * w)_{\mathbf{n}} = \sum_{\mathbf{k}\in\mathbb{Z}^d} v_{\mathbf{k}} w_{\mathbf{n}-\mathbf{k}}$. For each $h \in R_d$ let $\widetilde{h} \in \ell^2(\mathbb{Z}^d,\mathbb{R})$ be defined by $\widetilde{h}_{\mathbf{n}} = c_h(-\mathbf{n})$. If we use σ for the shift action of $\mathbb{Z}^d$ on $\ell^2(\mathbb{Z}^d,\mathbb{R})$, then the sign reversal in defining $\widetilde{h}$ means that $h(\sigma)v = \widetilde{h} * v$ for all $v \in \ell^2(\mathbb{Z}^d,\mathbb{R})$.

The Fourier transform sends each $v = (v_{\mathbf{n}}) \in \ell^2(\mathbb{Z}^d,\mathbb{R})$ to the function $\widehat{v}\colon \mathbb{T}^d \longrightarrow \mathbb{C}$ defined by

$$\widehat{v}(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} v_{\mathbf{n}} e^{2\pi i(\mathbf{t}\cdot\mathbf{n})},$$

where $\mathbf{t} = (t_1,\dots,t_d)$ and $\mathbf{t}\cdot\mathbf{n} = t_1 n_1 + \dots + t_d n_d$. By Plancherel's formula $\widehat{v} \in L^2(\mathbb{T}^d,\lambda_{\mathbb{T}^d})$, and we have $(\widetilde{h} * v)\widehat{\ } = \widehat{\widetilde{h}}\,\widehat{v}$ for $h \in R_d$ and $v \in \ell^2(\mathbb{Z}^d,\mathbb{R})$.

Let F denote the Fourier transform of $\widetilde{f}$. By Lemma 2.1(1), expansiveness of α implies that $F(\mathbf{t}) \neq 0$ for every $\mathbf{t} \in \mathbb{T}^d$, so that $1/F \in C^\infty(\mathbb{T}^d)$. Since

$$(1/F)(-\mathbf{t}) = \overline{1/F(\mathbf{t})},$$

the Fourier coefficients of $1/F$ are real. By applying the inverse Fourier transform to $1/F$ we obtain an element $w^\triangle \in \ell^2(\mathbb{Z}^d,\mathbb{R})$ with $\widehat{w^\triangle} = 1/F$ and

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} \|\mathbf{n}\|^k \, |w^\triangle_{\mathbf{n}}| < \infty$$

for all $k \geq 1$. (Indeed, $1/F$ is real-analytic, so that $w^\triangle_{\mathbf{n}}$ decays exponentially fast, but we do not need this stronger statement.) In particular, for each $h \in R_d$ the point $\widetilde{h} * w^\triangle = h(\sigma)w^\triangle \in \ell^2(\mathbb{Z}^d,\mathbb{R})$. Also note that $\widetilde{f} * w^\triangle$, as an element of $\ell^2(\mathbb{Z}^d,\mathbb{R})$, is the indicator function of $\{\mathbf{0}\}$.

Define $\eta\colon \ell^2(\mathbb{Z}^d,\mathbb{R}) \longrightarrow \mathbb{T}^{\mathbb{Z}^d}$ by reducing each coordinate (mod 1), and put $x^\triangle = \eta(w^\triangle)$. We call $x^\triangle$ the *fundamental homoclinic point of α*. The restriction of η to

$$\Delta = \{\, \widetilde{h} * w^{\Delta} : h \in R_d \,\}$$

is a group homomorphism of Δ into the homoclinic group $\Delta_\alpha(X)$ with the property that

$$\eta(\widetilde{h} * w^{\Delta}) = h(\sigma)(x^{\Delta})$$

for all $h \in R_d$. We claim that

$$\eta(\Delta) = \Delta_\alpha(X) \tag{4.1}$$

and

$$\{\, h \in R_d : \eta(\widetilde{h} * w^{\Delta}) = 0_X \,\} = fR_d. \tag{4.2}$$

In order to prove (4.1), let $x \in \Delta_\alpha(X)$. Denote the set of all elements in $\ell^2(\mathbb{Z}^d, \mathbb{R})$ with integral values by $\ell^2(\mathbb{Z}^d, \mathbb{Z})$. By Lemma 4.3 we can find $y \in \ell^2(\mathbb{Z}^d, \mathbb{R})$ with $\eta(y) = x$. Now $0 = f(\sigma)x = \eta(\widetilde{f} * y)$, so $\widetilde{f} * y \in \ell^2(\mathbb{Z}^d, \mathbb{Z})$. Since $y_{\mathbf{n}} \to 0$ as $\|\mathbf{n}\| \to \infty$ and $\widetilde{f}$ has finite support, it follows that $(\widetilde{f} * y)_{\mathbf{n}} \to 0$ as $\|\mathbf{n}\| \to \infty$. Since each $(\widetilde{f} * y)_{\mathbf{n}} \in \mathbb{Z}$, we see there is an $h \in R_d$ with $\widetilde{h} = \widetilde{f} * y$. Thus $\widehat{y} = \widehat{\widetilde{h}}/F$, so $y = \widetilde{h} * w^{\Delta}$ and $x = \eta(\widetilde{h} * w^{\Delta})$. This proves that $\eta(\Delta) = \Delta_\alpha(X)$, establishing (4.1).

For (4.2), observe that every $h \in R_d$ with $\eta(\widetilde{h} * w^{\Delta}) = h(\sigma)x^{\Delta} = 0$ satisfies that $\widetilde{h} * w^{\Delta} = \widetilde{g}$ for some $g \in R_d$, and by applying the inverse Fourier transform we see that $h = fg \in fR_d$. This completes the proof that the map $\tau(h) = \eta(\widetilde{h} * w^{\Delta})$ is an isomorphism from R_d/fR_d to $\Delta_\alpha(X)$.

To prove density of Δ_α, let $g \in \Delta_\alpha^{\perp}$. Then for every $h \in R_d$ we have that

$$1 = \langle g, \eta(\widetilde{h} * w^{\Delta}) \rangle = \exp\bigl[2\pi i (\widetilde{g} * \widetilde{h} * w^{\Delta})_{\mathbf{0}}\bigr] = \exp\bigl[2\pi i \bigl((\widetilde{g} * w^{\Delta}) * \widetilde{h}\bigr)_{\mathbf{0}}\bigr],$$

so that $\widetilde{g} * w^{\Delta} \in \ell^2(\mathbb{Z}^d, \mathbb{Z})$. Hence $\widetilde{g} * w^{\Delta} = \widetilde{k}$ for some $k \in R_d$, and $\widetilde{g} = \widetilde{g} * w^{\Delta} * \widetilde{f} = \widetilde{k} * \widetilde{f}$ implies that $g = kf \in fR_d$. It follows that $\Delta_\alpha^{\perp} = X^{\perp}$, so that Δ_α is dense in X. □

Example 4.6. (*Calculation of the fundamental homoclinic point for an expansive action.*) Let $f(u_1, u_2) = 3 - u_1 - u_2$. Clearly $V_{\mathbb{C}}(f) \cap \mathbb{S}^2 = \varnothing$, so that α_{R_2/fR_2} is expansive by Lemma 2.1(1). We compute the fundamental homoclinic point x^{Δ} as follows.

The Fourier transform F of $\widetilde{f}$ is $F(s,t) = 3 - e^{-2\pi i s} - e^{-2\pi i t}$. We can compute the inverse Fourier transform of $1/F$ explicitly using a geometric series:

$$\begin{aligned}\frac{1}{F(s,t)} &= \frac{1}{3}\left[\frac{1}{1 - \frac{1}{3}(e^{-2\pi i s} + e^{-2\pi i t})}\right] = \frac{1}{3}\sum_{k=0}^{\infty} 3^{-k}(e^{-2\pi i s} + e^{-2\pi i t})^k \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} 3^{-(m+n+1)}\binom{m+n}{n} e^{-2\pi i (ms+nt)}.\end{aligned}$$

Hence $w^{\triangle}$ is given by

$$w^{\triangle}_{(-m,-n)} = \begin{cases} 3^{-(m+n+1)}\binom{m+n}{n} & \text{if } m \geq 0 \text{ and } n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Every homoclinic point for α_{R_2/fR_2} is therefore an integral combination of translates of $x^{\triangle} = \eta(w^{\triangle})$, the reduction of $w^{\triangle}$ (mod 1).

A similar analysis applies to every polynomial in R_d having one coefficient whose absolute value strictly exceeds the sum of the absolute values of its other coefficients.

Example 4.7. (*Calculation of the fundamental homoclinic point for a hyperbolic toral automorphism.*) Let $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and let ϕ be the automorphism of $\mathbb{T}^2$ induced by A. The $\mathbb{Z}$-action generated by ϕ is an instance of our general algebraic framework as follows. Let $f(u_1) = u_1^2 - u_1 - 1$ and $\alpha = \alpha_{R_1/fR_1}$ be the corresponding $\mathbb{Z}$-action on $X = X_{R_1/fR_1} \subset \mathbb{T}^{\mathbb{Z}}$. Then $\psi\colon X \longrightarrow \mathbb{T}^2$ defined by $\psi(x) = (x_0, x_1)$ is easily checked to be an isomorphism of α with ϕ. In Example 3.3 we described $\Delta_\phi(\mathbb{T}^2)$ geometrically. Here we use the notations and proof of the previous lemma to compute the fundamental homoclinic point $x^{\triangle}$ of α analytically.

The Fourier transform of $\widetilde{f}$ is $F(t) = e^{-4\pi it} - e^{-2\pi it} - 1$. Let $\lambda = (1+\sqrt{5})/2$ and $\mu = (1-\sqrt{5})/2$ be the roots of f. Then

$$\frac{1}{F(t)} = \frac{1}{(e^{-2\pi it} - \lambda)(e^{-2\pi it} - \mu)} = \frac{1/\sqrt{5}}{e^{-2\pi it} - \lambda} - \frac{1/\sqrt{5}}{e^{-2\pi it} - \mu}.$$

Now

$$\frac{1/\sqrt{5}}{e^{-2\pi it} - \lambda} = -\frac{1}{\lambda\sqrt{5}}\,\frac{1}{1 - \lambda^{-1}e^{-2\pi it}} = \sum_{n\geq 0}\left(-\frac{1}{\sqrt{5}}\,\lambda^{-n-1}\right)e^{-2\pi int}$$
$$= \sum_{n\leq 0}\left(-\frac{1}{\sqrt{5}}\,\lambda^{n-1}\right)e^{2\pi int},$$

and

$$-\frac{1/\sqrt{5}}{e^{-2\pi it} - \mu} = -\frac{1}{\sqrt{5}\,e^{-2\pi it}}\,\frac{1}{1 - \mu e^{2\pi it}} = \sum_{n\geq 1}\left(-\frac{1}{\sqrt{5}}\,\mu^{n-1}\right)e^{2\pi int}.$$

Hence

$$w^{\triangle}_n = \begin{cases} -\dfrac{1}{\sqrt{5}}\,\mu^{n-1} & \text{if } n \geq 1, \\ -\dfrac{1}{\sqrt{5}}\,\lambda^{n-1} & \text{if } n \leq 0. \end{cases}$$

Using this one can verify directly the crucial property that

$$(\widetilde{f} * w^{\triangle})_n = w^{\triangle}_{n+2} - w^{\triangle}_{n+1} - w^{\triangle}_n = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $x^{\triangle} = \eta(w^{\triangle})$ is the fundamental homoclinic point of α.

To compare this result to the geometric construction in Example 3.3, recall the notation used there. It is then easy to compute that

$$(\mathcal{C} + (-1,0)) \cap \mathcal{E} = \{(w_0^{\triangle}, w_1^{\triangle})\} = \{\mathbf{x}\},$$

and so $\pi\mathbf{x} = \psi(x^{\triangle}) \in \Delta_\phi(\mathbb{T}^2)$.

This brings up a subtle point related to the remarks at the end of Example 3.3. Let $f(u_1) \in R_1$ be monic with constant term ± 1 and have degree k. Denote by A the transpose of the companion matrix of f. The matrix of multiplication by u_1 on $R_1/fR_1 \cong \mathbb{Z}^k$ with respect to the standard basis $\{1, u_1, \dots, u_1^{k-1}\}$ is $A^\top$, while in our formalism the matrix representing ϕ on $\mathbb{T}^k$ is A. Yet Lemma 4.5 implies that these matrices are conjugate in $GL(k,\mathbb{Z})$. The explanation for this discrepancy is as follows. There is an isomorphism $\theta\colon \mathbb{Z}^k \longrightarrow \Delta_\phi$ as in Example 3.3, and the matrix of ϕ on Δ_ϕ with respect to the image of the standard basis for $\mathbb{Z}^k$ under θ is indeed A. But Lemma 4.5 shows that if $t^{\triangle} \in \Delta_\phi$ is the fundamental homoclinic point for ϕ, then $\{t^{\triangle}, \phi(t^{\triangle}), \dots, \phi^{k-1}(t^{\triangle})\}$ is also a basis for Δ_ϕ, and with respect to this basis ϕ has matrix $A^\top$. It is this basis, not the standard one, that is used in the proof of Lemma 4.5. We are grateful to Manfred Einsiedler for pointing this out to us.

Proof of Theorem 4.1. Let α be an expansive algebraic $\mathbb{Z}^d$-action on X. If $\Delta_\alpha(X) \neq \{0_X\}$, then $h(\alpha) > 0$ by Lemma 4.4.

Conversely, suppose that $h(\alpha) > 0$. Then parts (1) and (3) of Lemma 2.1 guarantee that at least one prime ideal $\mathfrak{p}$ associated to the Noetherian R_d-module $M = \widehat{X}$ is principal and satisfies that $\alpha_{R_d/\mathfrak{p}}$ is mixing and expansive. By Lemma 2.1(5) we can find a Noetherian R_d-module $N \supset M$ with the properties described there, and we set $\beta = \alpha_N$ and $Y = X_N$. According to the description of N there exists a submodule $N' \subset N$ with $N/N' \cong R_d/\mathfrak{p}$, and by dualizing we obtain a β-invariant subgroup $Y' \subset Y$ with $Y' = \widehat{N/N'} \cong \widehat{R_d/\mathfrak{p}}$. If $\beta' \cong \alpha_{R_d/\mathfrak{p}}$ is the restriction of β to Y' then Lemma 4.5 shows that $\Delta_{\beta'}(Y')$ is dense in Y'. Lemma 2.1(3) shows that $h(\beta') = h(\alpha_{R_d/\mathfrak{p}}) > 0$, and Lemma 4.5 yields that $\{0_Y\} \neq \Delta_{\beta'}(Y') \subset \Delta_\beta(Y)$.

Let $Z \subset Y$ be the kernel of the surjective group homomorphism $\tau\colon Y \longrightarrow X$ dual to the inclusion $M \subset N$. Then Z is a closed, β-invariant subgroup of Y whose dual is N/M. Lemma 2.1(5) shows that $\beta = \alpha_N$ is expansive and so has finite entropy. The addition formula for entropy in [8, Appendix A] implies that the restriction β_Z of β to Z has entropy $h(\beta_Z) = h(\alpha_N) - h(\alpha_M) = 0$. Since β is expansive, so is β_Z, so that Lemma 4.4 shows that $\Delta_{\beta_Z}(Z) = \Delta_\beta(Y) \cap Z = \{0_Y\}$. Hence the restriction of $\tau\colon Y \longrightarrow X$ to $\Delta_\beta(Y)$ is injective. As $\Delta_\beta(Y) \neq \{0_Y\}$ and $\tau(\Delta_\beta(Y)) \subset \Delta_\alpha(X)$, we conclude that $\Delta_\alpha(X) \neq \{0_X\}$. □

We begin the proof of Theorem 4.2 with a definition.

Definition 4.8. Let $f \in R_d$ be an irreducible polynomial such that α_{R_d/fR_d} is expansive and mixing, let $k \geq 1$, and let

$$R = R_d/f^k R_d. \tag{4.3}$$

Suppose that $n \geq 1$ and define

$$[a,b] = \sum_{i=1}^{n} a_i b_i \in R \tag{4.4}$$

for every $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n) \in R^n$. If $N \subset R^n$ is an R-submodule we set

$$\begin{gathered} N' = \{b \in R^n : [a,b] = 0_R \text{ for every } a \in N\}, \\ N'' = (N')' = \{a \in R^n : [a,b] = 0_R \text{ for every } b \in N'\} \supset N. \end{gathered} \tag{4.5}$$

Lemma 4.9. *Let R be as in Definition 4.8, let $n \geq 1$, let $N \subset R^n$ be an R-module, and define the R-modules $N', N'' \subset R^n$ as in* (4.5). *We regard the R-modules $N \subset N'' \subset R^n$ as R_d-modules and consider the closed, shift-invariant subgroups*

$$\begin{gathered} X = X_{R^n} = \widehat{R^n} \subset \widehat{(R_d)^n} = (\mathbb{T}^n)^{\mathbb{Z}^d}, \\ Z = (N'')^{\perp} \subset Y = N^{\perp} \subset X \subset (\mathbb{T}^n)^{\mathbb{Z}^d}. \end{gathered}$$

Then the restrictions to X, Y and Z of the shift-action σ of $\mathbb{Z}^d$ on $(\mathbb{T}^n)^{\mathbb{Z}^d}$ are expansive, and the closure $\overline{\Delta_\sigma(Y)}$ of the homoclinic subgroup $\Delta_\sigma(Y) \subset Y$ is equal to Z. Furthermore there exists a group isomorphism $\tau \colon N' \longrightarrow \Delta_\sigma(Y)$ with $\sigma^{\mathbf{n}} \cdot \tau(a) = \tau(u^{\mathbf{n}} a)$ for every $a \in N'$ and $\mathbf{n} \in \mathbb{Z}^d$.

Proof. For $h \in R_d$ and $x \in X_R = \widehat{R} = (R_d/f^k R_d)^\frown \subset \mathbb{T}^{\mathbb{Z}^d}$ we define $h(\sigma)x \in X_R$ by (2.4) and observe that $h(\sigma)x = 0_{\mathbb{T}^{\mathbb{Z}^d}}$ whenever $x \in X_R$ and $h \in f^k R_d$. Hence we may abuse notation and set

$$a(\sigma)x = h(\sigma)x$$

for every $x \in X_R$ and $a = h + f^k R_d \in R$. With this convention we can write, for any submodule $L \subset R^n$, the group $L^{\perp} \subset X_{R^n} = (X_R)^n$ in the form

$$\begin{aligned} L^{\perp} = \Bigl\{ & x = (x^{(1)}, \dots, x^{(n)}) \in (X_R)^n : \\ & \sum_{i=1}^{n} a_i(\sigma) x^{(i)} = 0_{X_R} \text{ for every } (a_1, \dots, a_n) \in L \Bigr\}. \end{aligned} \tag{4.6}$$

According to Lemma 4.5, the homoclinic group $\Delta_\sigma(X)$ of the shift-action σ of $\mathbb{Z}^d$ on $X \subset (\mathbb{T}^n)^{\mathbb{Z}^d}$ is given by

$$\Delta_\sigma(X) = \{\, (h_1(\sigma)x^{\triangle}, \dots, h_n(\sigma)x^{\triangle}) : h = (h_1, \dots, h_n) \in R^n \,\}.$$

By (4.6), a homoclinic point $(h_1(\sigma)x^{\triangle}, \dots, h_n(\sigma)x^{\triangle}) \in \Delta_\sigma(X)$ lies in $N^{\perp} = Y$ if and only if

$$\sum_{i=1}^{n} a_i(\sigma) \cdot h_i(\sigma) x^{\triangle} = [a,h](\sigma) x^{\triangle} = 0_{X_R}$$

for every $a = (a_1, \dots, a_n) \in N$. From the proof of Lemma 4.5 we know that $g = 0_R$ is the only element in R having $g(\sigma)x^{\vartriangle} = 0_{X_R}$. Hence $(h_1(\sigma)x^{\vartriangle}, \dots, h_n(\sigma)x^{\vartriangle}) \in N^{\perp} = Y$ if and only if $[h, a] = 0_R$ for every $a \in N$ or, equivalently, if and only if $h \in N'$.

This shows that

$$\Delta_\sigma(Y) = \{ (h_1(\sigma)x^{\vartriangle}, \dots, h_n(\sigma)x^{\vartriangle}) : h = (h_1, \dots, h_n) \in N' \}$$

and establishes the promised isomorphism $\tau \colon N' \longrightarrow \Delta_\sigma(Y)$. Furthermore we see that an element $b = (b_1, \dots, b_n) \in R^n$ annihilates every σ-homoclinic point in Y if and only if $b \in N''$, and hence that $\overline{\Delta_\sigma(Y)} = Z$. □

The proof of the following lemma is based on ideas of P. Smith and R. Wiegand, which we use with their kind permission. Roughly speaking, this result would be easy if R were a field using a dimension argument, so we localize R at f to approximate a field and replace vector space dimension with module length.

Lemma 4.10. *Let R be as in Definition 4.8, let $n \geq 1$, and let $N \subset R^n$ be an R-module. Then there exists an element $g \in R \smallsetminus fR$ with $gN'' \subset N$.*

Proof. Let $S = R \smallsetminus fR$ be the semigroup of regular elements in R. Then $S \neq \varnothing$ since f is irreducible and hence not a unit by definition. Let $Q = S^{-1}R$ be the ring of fractions of R, which is the localization of R at f. Then every ideal of Q is of the form $\mathfrak{a}_j = \langle f^j \rangle = f^j Q$ for some $j \in \{1, \dots, k\}$, and

$$\{0_Q\} = \mathfrak{a}_k \subset \mathfrak{a}_{k-1} \subset \cdots \subset \mathfrak{a}_1 \subset Q.$$

Furthermore, if $\mathfrak{a} \subset Q$ is an ideal, and if $\gamma \colon \mathfrak{a} \longrightarrow Q$ is a Q-module homomorphism, then $\mathfrak{a} = \langle f^j \rangle$ for some $j \in \{1, \dots, k\}$, and

$$0_Q = \gamma(0_Q) = \gamma(f^{k-j} f^j) = f^{k-j}\gamma(f^j),$$

so that $\gamma(f^j) = g f^j$ for some $g \in Q$. By setting $\bar{\gamma}(h) = gh$ for every $h \in Q$ we have extended $\gamma \colon \mathfrak{a} \longrightarrow Q$ to a Q-module homomorphism $\bar{\gamma} \colon Q \longrightarrow Q$.

We claim that, if $M_1 \subset M_2$ are Q-modules, then every Q-module homomorphism $\theta \colon M_1 \longrightarrow Q$ can be extended to a Q-module homomorphism $\bar{\theta} \colon M_2 \longrightarrow Q$ (i.e. that Q is injective). Indeed, let $\overline{M}_1 \subset M_2$ be a maximal Q-module for which there exists such an extension $\bar{\theta} \colon \overline{M}_1 \longrightarrow Q$ of θ. If $\overline{M}_1 \neq M_2$, then we choose $a \in M_2 \smallsetminus \overline{M}_1$, set $\mathfrak{a} = \{h \in Q : h \cdot a \in \overline{M}_1\}$, define a Q-module homomorphism $\gamma \colon \mathfrak{a} \longrightarrow Q$ by $\gamma(h) = \bar{\theta}(h \cdot a)$, and apply the above observation to find an extension $\bar{\gamma} \colon Q \longrightarrow Q$ of γ. Then the Q-module homomorphism $\bar{\theta}' \colon \overline{M}_1 + Q \cdot a \longrightarrow Q$, defined by $\bar{\theta}'(b + h \cdot a) = \bar{\theta}(b) + \bar{\gamma}(h)$ for every $h \in Q$ and $b \in \bar{M}_1$, is a proper extension of $\bar{\theta}$. This contradiction implies that θ can indeed be extended to all of M_2.

For every Q-module M we denote by $M^* = \operatorname{Hom}_Q(M, Q)$ the set of all Q-module homomorphism $b \colon M \longrightarrow Q$. Then M^* is a Q-module with respect to the operation $(h, b) \mapsto h \cdot b$, defined by $(h \cdot b)(a) = b(h \cdot a)$ for every $h \in Q$, $b \in M^*$ and $a \in M$. The module M^* is called the *dual* module of M. In this terminology we can rephrase the above extension property by saying that, for every short exact sequence

$$0 \longrightarrow M_1 \xrightarrow{\iota} M_2 \longrightarrow M_3 \longrightarrow 0$$

of Q-modules, where ι is the inclusion map, the sequence

$$0 \longrightarrow M_3^* \longrightarrow M_2^* \xrightarrow{\iota^*} M_1^* \longrightarrow 0 \tag{4.7}$$

is exact, where ι^* is the restriction map.

For every Q-module K and nonzero $a \in K$ the ideal $\mathfrak{a} = \operatorname{ann}(a) = \{h \in Q : h \cdot a = 0_Q\}$ is of the form $\mathfrak{a} = \langle f^j \rangle$ for some $j \in \{1, \dots, k\}$, and by setting $b = f^{j-1} \cdot a$ we obtain a nonzero element in K whose annihilator satisfies that $\operatorname{ann}(b) = \langle f \rangle$. It follows that there exists, for every nonzero Noetherian Q-module K, a filtration

$$\{0_K\} = K_0 \subset K_1 \subset \cdots \subset K_l = K$$

such that $K_j/K_{j-1} \cong Q/\langle f \rangle$ for every $j = 1, \dots, l$. The integer $l = l(K) \geq 1$ is independent of the specific filtration chosen and is called the *length* of K. From the independence of $l(K)$ of the specific filtration we conclude that if

$$0 \longrightarrow K_1 \longrightarrow K_2 \longrightarrow K_3 \longrightarrow 0$$

is a short exact sequence of Noetherian Q-modules, then

$$l(K_2) = l(K_1) + l(K_3). \tag{4.8}$$

If $l(K) = 1$ then $K \cong Q/\langle f \rangle \cong K^*$, and $l(K^*) = l(K) = 1$. Repeated application of (4.7) and (4.8) shows that $l(K) = l(K^*)$ for every Noetherian Q-module K.

Consider the Q-module $M = Q \otimes_R N \subset Q \otimes_R R^n = Q^n$, define the Q-modules $M', M'' \subset Q^n$ as in (4.5), and observe that $M' = Q \otimes_R N'$ and $M'' = Q \otimes_R N''$. By applying the above discussion to $M \subset M''$ and M' we see that the sequences

$$\begin{gathered} 0 \longrightarrow M \longrightarrow Q^n \longrightarrow L \longrightarrow 0, \\ 0 \longrightarrow L^* \longrightarrow (Q^n)^* \longrightarrow M^* \longrightarrow 0 \end{gathered} \tag{4.9}$$

are exact, where $L = Q^n/M$, $(Q^n)^* \cong Q^n$ and $L^* = M'$. By replacing M with M' in the second exact sequence in (4.9) we obtain that the sequence

$$0 \longrightarrow M'' \longrightarrow Q^n \longrightarrow (M')^* \longrightarrow 0 \tag{4.10}$$

is again exact, and that

$$kn = l(Q^n) = l(M) + l(L) = l(L^*) + l(M^*) = l(M') + l(M) = l(M'') + l(M').$$

Hence $l(M) = l(M'')$, and therefore $M = M''$, since $M \subset M''$, $l(M'') = l(M) + l(M''/M)$, and $l(M''/M) = 0$.

We have proved that

$$\{0\} = M''/M = (Q \otimes_R N'')/(Q \otimes_R N) = Q \otimes_R (N''/N),$$

i.e. that there exists, for every $a \in N''$, an element $g \in R \smallsetminus fR$ with $g \cdot a \in N$. As N'' is Noetherian we can find a single element $g \in R \smallsetminus fR$ with $g \cdot N'' \subset N$, as claimed. □

Lemma 4.11. *Assume the hypotheses of Lemma 4.9 and also that* $Y \neq \{0_X\}$. *Then the following conditions are equivalent.*

(1) *The restriction of the shift-action* σ *of* $\mathbb{Z}^d$ *on* $\widehat{R^n} \subset (\mathbb{T}^n)^{\mathbb{Z}^d}$ *to* $Y = N^{\perp}$ *has completely positive entropy;*
(2) $\Delta_\sigma(Y)$ *is dense in* Y*;*
(3) $N'' = N$.

Proof. (1) ⇒ (2) ⇒ (3): Regard $N, N'' \subset R^n$ as R_d-modules. Then $\widehat{Y/Z} = (R^n/N)/(R^n/N'') \cong N''/N$, so that the $\mathbb{Z}^d$-action $\sigma_{Y/Z}$ induced by the shift-action σ on X is isomorphic to $\alpha_{N''/N}$. Lemma 4.10 implies that there is a nonunit g such that $gN'' \subset N$, so that each prime ideal associated with the R_d-module N''/N is nonprincipal since it contains both f and an irreducible factor of g. Then Lemma 2.1(3) shows that $h(\alpha_{N''/N}) = 0$. Since σ_Y has completely positive entropy, the factor $\sigma_{Y/Z}$ must be trivial. Hence $Z = \overline{\Delta_\sigma(Y)} = Y$ and $N'' = N$.

(2) ⇒ (1): Since $Y \neq \{0_X\}$ and $\Delta_\sigma(Y)$ is dense in Y, then the entropy $h(\sigma_Y)$ of the shift-action σ_Y of $\mathbb{Z}^d$ on Y is positive by Lemma 4.4. If σ_Y does not have completely positive entropy, then Theorem 6.4 of [8] implies there would exist a closed, shift-invariant proper subgroup $K \subset Y$ with $h(\sigma_{Y/K}) = 0$. Since $\sigma_{Y/K}$ would be expansive (see [14, Cor. 3.11]), Lemma 4.4 would then imply that $\Delta_\sigma(Y) \subset K$, contradicting (2). Hence (2) ⇒ (1).

(3) ⇒ (2): This follows from Lemma 4.9. □

Proof of Theorem 4.2. Let α be an expansive and mixing algebraic $\mathbb{Z}^d$-action on X. If $\Delta_\alpha(X)$ is nontrivial and dense in X then we see exactly as in the proof of the implication (2) ⇒ (1) in Lemma 4.11 that α has completely positive entropy.

Conversely, if α has completely positive entropy we denote by $M = \widehat{X}$ the Noetherian R_d-module defined by (2.1)–(2.2), write $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ for the set of associated prime ideals of M, and note that, by Lemma 2.1, $\mathfrak{p}_j$ is principal and $\alpha_{R_d/\mathfrak{p}_j}$ expansive and mixing for every $j = 1, \dots, m$. We choose irreducible Laurent polynomials $f_1, \dots, f_m \in R_d$ with $\mathfrak{p}_j = \langle f_j \rangle = f_j R_d$ for $j = 1, \dots, m$ and write $N = N^{(1)} \oplus \dots \oplus N^{(m)} \supset M$ for the Noetherian R_d-module appearing in Lemma 2.1(5). If we can prove that $\Delta_{\alpha_{N^{(j)}}}(X_{N^{(j)}})$ is dense in $X_{N^{(j)}}$ for every $j = 1, \dots, n$, then

$$\Delta_{\alpha_N}(X_N) = \prod_{j=1}^{m} \Delta_{\alpha_{N^{(j)}}}(X_{N^{(j)}})$$

is dense in X_N, and the surjective group homomorphism $\tau \colon X_N \longrightarrow X = X_M$ dual to the inclusion $M \subset N$ satisfies that $\tau \cdot \alpha_N^{\mathbf{n}} = \alpha^{\mathbf{n}} \cdot \tau$ for every $\mathbf{n} \in \mathbb{Z}^d$, and that $\tau(\Delta_{\alpha_N}(X_N)) \subset \Delta_\alpha(X)$. In particular, $\Delta_\alpha(X)$ is dense in X.

We fix $j \in \{1, \dots, m\}$ and recall from Lemma 2.1(5) that $N^{(j)}$ has a filtration

$$N^{(j)} = N^{(j)}_{s_j} \supset \cdots \supset N^{(j)}_0 = \{0\} \tag{4.11}$$

with $N^{(j)}_l / N^{(j)}_{l-1} \cong R_d / f_j R_d$ for every $l = 1, \dots, s_j$. Since $\mathfrak{p}_j = \langle f_j \rangle$ is the only prime ideal associated with $N^{(j)}$, it follows that $\alpha_{N^{(j)}}$ is mixing and has completely positive entropy by parts (2) and (4) of Lemma 2.1. The filtration (4.11) allows us to regard $N^{(j)}$ as a module over the ring $R = R_d / f_j^{s_j} R_d$, since $f_j^{s_j} \cdot a = 0_{N^{(j)}}$ for every $a \in N^{(j)}$.

Choose elements $a_1, \dots, a_n$ in $N^{(j)}$ with $N^{(j)} = R \cdot a_1 + \cdots + R \cdot a_n$ and consider the surjective R-module homomorphism $\kappa \colon R^n \longrightarrow N^{(j)}$ defined by $\kappa(h_1, \dots, h_n) = h_1 \cdot a_1 + \cdots + h_n \cdot a_n$ for every $(h_1, \dots, h_n) \in R^n$. If K is the kernel of κ, then $K \subset R^n$ is an R-submodule (and hence also an R_d-submodule) of R^n, and

$$X_{N^{(j)}} = K^{\perp} \subset \widehat{R^n} \subset \widehat{(R_d)^n} = (\mathbb{T}^n)^{\mathbb{Z}^d}.$$

As $\alpha_{N^{(j)}}$ has completely positive entropy, it follows from Lemma 4.11 that $\Delta_{\alpha_{N^{(j)}}}(X_{N^{(j)}})$ is dense in $X_{N^{(j)}} = K^{\perp}$, as claimed. This completes the proof of Theorem 4.2. □

Remark 4.12. Let α be an expansive algebraic $\mathbb{Z}^d$-action on X. The previous proof shows that $Z = \overline{\Delta_\alpha(X)}$ is the maximal closed α-invariant subgroup of X on which α has completely positive entropy. This gives an intrinsic description of the Pinsker algebra of α as the inverse image of the Borel σ-algebra of X/Z under the quotient map $X \longrightarrow X/Z$ (cf. [8, §6]). Hence despite the uncomplicated dynamics of individual homoclinic points in X, their closure is precisely the largest closed invariant subgroup of X on which the action is Bernoulli (see [15, Thm. 23.1]).

5. Specification

Specification is an orbit tracing property that has proved useful in the study of expansive homeomorphisms. Ruelle [12] investigated the extension of this notion to $\mathbb{Z}^d$-actions, motivated by statistical mechanics. Our purpose in this section is to show that expansive algebraic $\mathbb{Z}^d$-actions provide a large class of $\mathbb{Z}^d$-actions having specification.

First recall the definition of the norm $\|\cdot\|$ on $\mathbb{Z}^d$ from (2.10) and of the cube $\mathcal{B}(r) \subset \mathbb{Z}^d$ from (2.11).

Definition 5.1. (1) Let T be a continuous $\mathbb{Z}^d$-action on a compact metric space (X, ρ). The action T has *weak specification* if there exists, for every $\varepsilon > 0$, an integer $p(\varepsilon) \geq 1$ with the following property: for every finite collection $\mathcal{Q}_1, \dots, \mathcal{Q}_t$ of rectangles $\mathcal{Q}_j = \prod_{i=1}^d \{a_i, \dots, b_i\} \subset \mathbb{Z}^d$ with

$$\operatorname{dist}(\mathcal{Q}_j, \mathcal{Q}_k) = \min_{\mathbf{m} \in \mathcal{Q}_j, \, \mathbf{n} \in \mathcal{Q}_k} \|\mathbf{m} - \mathbf{n}\| \geq p(\varepsilon) \quad \text{for } 1 \leq j < k \leq t, \tag{5.1}$$

and for every collection of points $x^{(1)}, \dots, x^{(t)}$ in X, there exists a point $y \in X$ with

$$\rho(T^{\mathbf{n}}y, T^{\mathbf{n}}x^{(j)}) < \varepsilon \quad \text{for all } \mathbf{n} \in \mathcal{Q}_j,\ 1 \le j \le t. \tag{5.2}$$

(2) The $\mathbb{Z}^d$-action T has *strong specification* if there exists, for every $\varepsilon > 0$, an integer $p(\varepsilon) \ge 1$ with the following property: for every collection of rectangles $\mathcal{Q}_1, \dots, \mathcal{Q}_t$ in $\mathbb{Z}^d$ satisfying (5.1) and every subgroup $\Gamma \subset \mathbb{Z}^d$ with

$$\operatorname{dist}(\mathcal{Q}_j + \mathbf{q}, \mathcal{Q}_k) = \min_{\mathbf{m} \in \mathcal{Q}_j + \mathbf{q},\, \mathbf{n} \in \mathcal{Q}_k} \|\mathbf{m} - \mathbf{n}\| \ge p(\varepsilon) \tag{5.3}$$

whenever $1 \le j, k \le t$ and $\mathbf{q} \in \Gamma \smallsetminus \{\mathbf{0}\}$, and for every collection of points $x^{(1)}, \dots, x^{(t)}$ in X, there exists a point $y \in X$ satisfying (5.2) and with

$$T^{\mathbf{m}}y = y$$

for every $\mathbf{m} \in \Gamma$.

(3) Let α be an algebraic $\mathbb{Z}^d$-action on X with homoclinic group $\Delta_\alpha(X)$, and let ρ be a metric on X consistent with its topology. The action α has *homoclinic specification* if there exists, for every $\varepsilon > 0$, an integer $p(\varepsilon) \ge 1$ with the following property: for every rectangle $\mathcal{Q} \subset \mathbb{Z}^d$ and every $x \in X$ there exists an α-homoclinic point $y \in \Delta_\alpha(X)$ with

$$\rho(\alpha^{\mathbf{n}}x, \alpha^{\mathbf{n}}y) < \varepsilon \ \text{ for all } \mathbf{n} \in \mathcal{Q},$$
$$\rho(0_X, \alpha^{\mathbf{n}}y) < \varepsilon \ \text{ for all } \mathbf{n} \in \mathbb{Z}^d \smallsetminus (\mathcal{Q} + \mathcal{B}(p(\varepsilon))).$$

Note that each of these three properties is preserved under taking quotients.

This section is devoted to the proof of the following theorem.

Theorem 5.2. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on a nontrivial compact abelian group X. Then the following are equivalent.*

(1) *α has completely positive entropy;*
(2) *α has weak specification;*
(3) *α has strong specification;*
(4) *α has homoclinic specification.*

The proof of this requires three lemmas. For the first of these we assume that $f \in R_d$ is a (possibly reducible) polynomial such that the $\mathbb{Z}^d$-action $\alpha = \alpha_{R_d/fR_d}$ on $X = X_{R_d/fR_d}$ is expansive. As in §2 we view X as the closed, shift-invariant subgroup

$$X = \{x \in \mathbb{T}^{\mathbb{Z}^d} : f(\sigma)x = 0_{\mathbb{T}^{\mathbb{Z}^d}}\}$$

and identify α with the restriction σ_X of the shift-action σ on $\mathbb{T}^{\mathbb{Z}^d}$ to X. If σ also denotes the $\mathbb{Z}^d$-shift on $\ell^\infty(\mathbb{Z}^d, \mathbb{R})$, then for every $h \in R_d$ and $w \in \ell^\infty(\mathbb{Z}^d, \mathbb{R})$ the point $h(\sigma)w = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_h(\mathbf{n})\sigma^{\mathbf{n}}w \in \ell^\infty(\mathbb{Z}^d, \mathbb{R})$. If we define $\widetilde{h}$ as in the proof of Lemma 4.5, then $h(\sigma)w = \widetilde{h} * w$. Let $\ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \subset \ell^\infty(\mathbb{Z}^d, \mathbb{R})$ be the subgroup of bounded, integer-valued maps from $\mathbb{Z}^d$ to

$\mathbb{Z}$ and let $\eta\colon \ell^\infty(\mathbb{Z}^d,\mathbb{R}) \longrightarrow \mathbb{T}^{\mathbb{Z}^d}$ be the map that reduces each coordinate (mod 1). Then

$$\eta^{-1}(X) = \{\, v \in \ell^\infty(\mathbb{Z}^d,\mathbb{R}) : f(\sigma)v \in \ell^\infty(\mathbb{Z}^d,\mathbb{Z}) \,\}.$$

Let $\|\cdot\|_\infty$ denote the supremum norm on $\ell^\infty(\mathbb{Z}^d,\mathbb{R})$, $|\cdot|\colon \mathbb{T} \longrightarrow \mathbb{R}$ be defined by (2.9), and write $w \in \ell^\infty(\mathbb{Z}^d,\mathbb{R})$ as $w = (w_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d)$.

Lemma 5.3. *For every $\varepsilon > 0$ and $L > 0$ there is an $r > 0$ such that if $u, v \in \eta^{-1}(X)$, $\|u\|_\infty \le L$, $\|v\|_\infty \le L$, and $(f(\sigma)u)_{\mathbf{n}} = (f(\sigma)v)_{\mathbf{n}}$ for all $\mathbf{n} \in \mathcal{B}(r)$, then $|\eta(u)_{\mathbf{0}} - \eta(v)_{\mathbf{0}}| < \varepsilon$. Consequently, if $u, v \in \eta^{-1}(X)$ and $f(\sigma)u = f(\sigma)v$, then $u = v$.*

Proof. Suppose that for some $\varepsilon > 0$ and $L > 0$ we can find, for every $r > 0$, elements $u^{(r)}$, $v^{(r)} \in \eta^{-1}(X)$ with $\|u^{(r)}\|_\infty \le L$, $\|v^{(r)}\|_\infty \le L$, $(f(\sigma)u^{(r)})_{\mathbf{n}} = (f(\sigma)v^{(r)})_{\mathbf{n}}$ for all $\mathbf{n} \in \mathcal{B}(r)$, and $|\eta(u^{(r)})_{\mathbf{0}} - \eta(v^{(r)})_{\mathbf{0}}| \ge \varepsilon$. Then the sequence $\{u^{(r)} - v^{(r)} : r = 1, 2, \dots\}$ has a limit point $w \in \ell^\infty(\mathbb{Z}^d,\mathbb{R})$ in the weak*-topology (i.e. the topology of pointwise convergence) such that $|w_{\mathbf{0}}| \ge \varepsilon$ and $f(\sigma)w = 0$. Hence $\eta(tw) \in X$ for all $t \in \mathbb{R}$, and by choosing t sufficiently small we have that $\eta(tw) \ne 0_X$ and $\sup_{\mathbf{n}\in\mathbb{Z}^d} |\eta(tw)_{\mathbf{n}}|$ is arbitrarily small, contradicting expansiveness of $\alpha = \sigma_X$.

The second assertion follows easily from the first by taking limits. □

Lemma 5.4. *If α_{R_d/fR_d} is expansive then it has both homoclinic and strong specification.*

Proof. As in the proof of Lemma 4.5 we regard X as a closed, shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}^d}$ and α as the restriction σ_X to X of the shift-action σ of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$. The proof of Lemma 4.5 shows that $\Delta_\alpha(X) = \Delta_{\sigma_X}(X)$ is dense in X, and that every element $y \in \Delta_\alpha(X)$ is of the form $y = h(\sigma)x^\triangle$ for some $h \in R = R_d/fR_d$. Here $x^\triangle = \eta(w^\triangle)$, where $w^\triangle \in \ell^\infty(\mathbb{Z}^d,\mathbb{R})$ is the unique point in $\eta^{-1}(X)$ with $(f(\sigma)w^\triangle)_{\mathbf{0}} = 1$ and $(f(\sigma)w^\triangle)_{\mathbf{n}} = 0$ for all $\mathbf{n} \in \mathbb{Z}^d \smallsetminus \{\mathbf{0}\}$, i.e., $\widetilde{f} * w^\triangle$ is the convolutional identity for $\ell^\infty(\mathbb{Z}^d,\mathbb{R})$.

By Lemma 4.3 we have that

$$\|w^\triangle\|_1 = \sum_{\mathbf{n}\in\mathbb{Z}^d} |w^\triangle_{\mathbf{n}}| < \infty.$$

Write $f = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n})$ and put

$$\|f\| = \sum_{\mathbf{n}\in\mathbb{Z}^d} |c_f(\mathbf{n})|.$$

Let $L = \max\{1, \|w^\triangle\|_1 \cdot \|f\|\}$.

Fix $\varepsilon > 0$. Apply Lemma 5.3 with this choice of ε and L to find an integer $r > 0$ such that every pair of points $u, v \in \eta^{-1}(X) \subset \ell^\infty(\mathbb{Z}^d,\mathbb{R})$ with $\|u\|_\infty \le L$, $\|v\|_\infty \le L$, and $(f(\sigma)u)_{\mathbf{n}} = (f(\sigma)v)_{\mathbf{n}}$ for every $\mathbf{n} \in \mathcal{B}(r)$ satisfies that $|\eta(u)_{\mathbf{0}} - \eta(v)_{\mathbf{0}}| < \varepsilon$. It follows that if $\mathcal{Q} \subset \mathbb{Z}^d$ is a rectangle, and if $u, v \in \eta^{-1}(X)$ satisfy that $\|u\|_\infty \le L$, $\|v\|_\infty \le L$, and $(f(\sigma)u)_{\mathbf{n}} = (f(\sigma)v)_{\mathbf{n}}$

for every $\mathbf{n} \in \mathcal{Q}+\mathcal{B}(r)$, then $|\eta(u)_{\mathbf{n}}-\eta(v)_{\mathbf{n}}| < \varepsilon$ for all $\mathbf{n} \in \mathcal{Q}$. By increasing r if necessary, Lemma 4.3 allows us to assume in addition that

$$\sum_{\mathbf{n}\in\mathbb{Z}^d\smallsetminus\mathcal{B}(r)} |w^{\triangle}_{\mathbf{n}}| < \varepsilon/\|f\|.$$

Take an arbitrary element $x \in X$, choose $u \in \ell^\infty(\mathbb{Z}^d,\mathbb{R})$ with $\|u\|_\infty \le 1$ and $\eta(u) = x$, and consider the point $z = f(\sigma)u \in \ell^\infty(\mathbb{Z}^d,\mathbb{Z})$. If $\mathcal{Q} \subset \mathbb{Z}^d$ is a fixed rectangle, define $h = \sum_{\mathbf{m}\in\mathbb{Z}^d} c_h(\mathbf{m})u^{\mathbf{m}} \in R_d$ by

$$c_h(\mathbf{m}) = \begin{cases} z_{\mathbf{m}} & \text{if } \mathbf{m} \in \mathcal{Q}+\mathcal{B}(r), \\ 0 & \text{otherwise.} \end{cases}$$

Let $v = h(\sigma)w^{\triangle}$ and put $y = h(\sigma)x^{\triangle} = \eta(\widetilde{h} * w^{\triangle}) \in \Delta_\alpha(X)$. Since $\widetilde{f} * w^{\triangle}$ is the convolutional identity in $\ell^\infty(\mathbb{Z}^d,\mathbb{R})$, it follows that $f(\sigma)v = f(\sigma)h(\sigma)w^{\triangle} = \widetilde{h} * \widetilde{f} * w^{\triangle} = \widetilde{h}$. Hence

$$\big(f(\sigma)v\big)_{\mathbf{n}} = \big(f(\sigma)h(\sigma)w^{\triangle}\big)_{\mathbf{n}} = c_h(\mathbf{n}) = \big(f(\sigma)u\big)_{\mathbf{n}}$$

for every $\mathbf{n} \in \mathcal{Q}+\mathcal{B}(r)$ and

$$\big(f(\sigma)v\big)_{\mathbf{n}} = \big(f(\sigma)h(\sigma)w^{\triangle}\big)_{\mathbf{n}} = 0$$

for every $\mathbf{n} \in \mathbb{Z}^d \smallsetminus (\mathcal{Q}+\mathcal{B}(r))$. Our choice of r implies that

$$|x_{\mathbf{n}} - y_{\mathbf{n}}| < \varepsilon \text{ for every } \mathbf{n} \in \mathcal{Q},$$
$$|y_{\mathbf{n}}| < \varepsilon \text{ for every } \mathbf{n} \in \mathbb{Z}^d \smallsetminus (\mathcal{Q}+\mathcal{B}(2r)).$$

This is easily seen to imply the homoclinic specification of α.

To check strong specification for α we assume that ε, L, and r are chosen as above, and set $p(\varepsilon) = 2r$. Consider finitely many points $x^{(1)},\dots,x^{(t)}$ in X, rectangles $\mathcal{Q}_1,\dots,\mathcal{Q}_t$ in $\mathbb{Z}^d$, and a subgroup $\Gamma \subset \mathbb{Z}^d$ satisfying (5.1) and (5.3). For each of these rectangles $\mathcal{Q}_j$ we find a homoclinic point $y^{(j)}$ with

$$|x^{(j)}_{\mathbf{n}} - y^{(j)}_{\mathbf{n}}| < \varepsilon \text{ for every } \mathbf{n} \in \mathcal{Q}_j,$$
$$\sum_{\substack{i=1,\dots,t\\ i\ne j}} |y^{(i)}_{\mathbf{n}}| + \sum_{i=1}^{t} \sum_{\mathbf{0}\ne\mathbf{m}\in\Gamma} |y^{(i)}_{\mathbf{n}+\mathbf{m}}| < \varepsilon \text{ for every } \mathbf{n} \in \mathbb{Z}^d \smallsetminus (\mathcal{Q}^{(j)}+\mathcal{B}(2r)).$$

The point

$$y = \sum_{j=1}^{t} \sum_{\mathbf{m}\in\Gamma} \alpha^{\mathbf{m}}(y^{(j)})$$

satisfies (5.2) with ε replaced by 2ε, and $\alpha^{\mathbf{m}}y = y$ for every $\mathbf{m} \in \Gamma$. □

Lemma 5.5. *Let $f \in R_d$ be an irreducible polynomial such that α_{R_d/fR_d} is expansive, $k,n \ge 1$, $R = R_d/f^kR_d$, and $X \subset \widehat{R^n} \subset (\mathbb{T}^n)^{\mathbb{Z}^d}$ be a closed, shift-invariant subgroup. If the restriction to X of the shift-action σ of $\mathbb{Z}^d$ on $(\mathbb{T}^n)^{\mathbb{Z}^d}$ has completely positive entropy, then it has both homoclinic and strong specification.*

Proof. We write N for the R_d-module $X^{\perp}$, considered as an R-module, and apply the Lemmas 4.9 and 4.11 to conclude that $N'' = N$, that $\Delta_\sigma(X)$ is dense in X, and that every homoclinic point $y \in \Delta_\sigma(X) \subset \widehat{R^n} \subset (\mathbb{T}^{\mathbb{Z}^d})^n \cong (\mathbb{T}^n)^{\mathbb{Z}^d}$ is of the form

$$y = (h_1(\sigma)x^{\triangle}, \dots, h_n(\sigma)x^{\triangle})$$

for some $(h_1, \dots, h_n) \in N' \subset R^n$ (the notation is explained in the proof of Lemma 4.9).

For each $h = (h_1, \dots, h_n) \in N'$ and $y \in X_R$ the element $\zeta_h(y) = (h_1(\sigma)y, \dots, h_n(\sigma)y)$ is in X since it is annihilated by every element in $N = X^{\perp}$. Choose a finite set $\{h^{(i)} : 1 \le i \le s\}$ of generators for N' and consider the map $\zeta\colon X_R^s \longrightarrow X$ defined by $\zeta(y_1, \dots, y_s) = \sum_{i=1}^s \zeta_{h^{(i)}}(y_i)$. Let $Y = \zeta(X_R^s)$. Then clearly $Y^{\perp} \supset N'' = N$. Conversely, if $b = (b_1, \dots, b_n) \in Y^{\perp}$, then b annihilates each $\zeta_{h^{(i)}}(x^{\triangle})$, so $b \in X^{\perp} = N$ since $\Delta_\sigma(X)$ is dense in X. Hence $Y = X$ and ζ is surjective.

Lemma 5.4 shows that α_R satisfies both homoclinic and strong specification. Since each property is preserved under finite direct products and quotients, we conclude that σ_X also satisfies homoclinic and strong specification. □

Proof of Theorem 5.2. (1) ⇒ (2), (3), and (4): Let α be an expansive algebraic $\mathbb{Z}^d$-action on X with completely positive entropy, and let $M = \widehat{X}$ be the Noetherian R_d-module defined in (2.1)–(2.2). We choose an R_d-module $N = N^{(1)} \oplus \dots \oplus N^{(m)} \supset M$ according to Lemma 2.1(5) and put $Y = X_N$ and $\beta = \alpha_N$. Fix $j \in \{1, \dots, m\}$, denote by f_j a generator of the principal prime ideal $\mathfrak{p}_j$ associated with M, and view the module $N^{(j)}$ as a module over the ring $R = R_d/f_j^{s_j} R_d$ (cf. the proof of Theorem 4.2). We continue as in the proof of Theorem 4.2, write $N^{(j)} = R^{s_j}/K$ for some R-submodule $K \subset R^{s_j}$, and embed $X_{N^{(j)}}$ in $X_{R^{s_j}} \subset (\mathbb{T}^{s_j})^{\mathbb{Z}^d}$ as a closed, shift-invariant subgroup. As $\alpha_{N^{(j)}}$ has completely positive entropy, it has homoclinic and strong specification by Lemma 5.5, and by varying $j \in \{1, \dots, m\}$ we obtain that $\beta = \alpha_N$ has homoclinic and strong specification.

If $\tau\colon Y \longrightarrow X$ is the surjective group homomorphism dual to the inclusion $M \subset N$, then $\tau \cdot \beta_{\mathbf{n}} = \alpha^{\mathbf{n}} \cdot \tau$, and hence α has both homoclinic and strong specification since each property is preserved under quotients.

(3) ⇒ (2) ⇒ (1): Trivially (3) ⇒ (2). Suppose that α is expansive and has weak specification. Let $Z = \overline{\Delta_\alpha(X)}$. Then by Remark 4.12, the restriction $\alpha_{X/Z}$ of α to X/Z has zero entropy. Now weak specification implies positive entropy and is preserved under quotients. Hence if X/Z is nontrivial then we would have $h(\alpha_{X/Z}) > 0$, a contradiction. Thus $Z = X$, and so α has completely positive entropy by Theorem 4.2.

(4) ⇒ (1): By Proposition 2.2, if α has homoclinic specification then $\Delta_\alpha(X)$ is dense in X, hence α has completely positive entropy by Theorem 4.2. □

Remark 5.6. Definition 5.1 uses rectangles $\mathcal{Q}_j$ in $\mathbb{Z}^d$ in defining specification. However, our proofs made no essential use of the particular form of these sets, only their separation described in (5.1). This leads to a very strong form of specification. Namely, every expansive algebraic $\mathbb{Z}^d$-action with completely positive entropy satisfies parts (1), (2), and (3) of Definition 5.1 where the sets $\mathcal{Q}_j$ can be *arbitrary* subsets of $\mathbb{Z}^d$, finite or infinite.

One simple consequence is the following. Let α be an expansive algebraic $\mathbb{Z}^d$-action on (X,ρ) and $\epsilon > 0$. Then there is a $p(\epsilon) > 0$ such that for every $r > 0$ and $x, y \in X$, there is a $z \in X$ such that $\rho(\alpha^{\mathbf{n}} z, \alpha^{\mathbf{n}} x) < \epsilon$ if $\|\mathbf{n}\| < r$ and $\rho(\alpha^{\mathbf{n}} z, \alpha^{\mathbf{n}} y) < \epsilon$ if $\|\mathbf{n}\| > r + p(\epsilon)$. This property does not appear to be a direct consequence of Theorem 5.2.

Remark 5.7. Theorem 5.2 can fail if the expansiveness hypothesis is omitted. For example, let ϕ be an ergodic automorphism of $\mathbb{T}^k$ induced by $A \in GL(k,\mathbb{Z})$. Then ϕ has completely positive entropy. On the other hand, the argument of Example 3.4 shows that $\Delta_\phi(\mathbb{T}^k) = \{\mathbf{0}\}$, so that ϕ does not have homoclinic specification. Furthermore, ϕ has strong specification if and only if A has no eigenvalues on the unit circle, and ϕ has weak specification if and only if all eigenvalues of A on the unit circle are semisimple (see [7]).

In Examples 3.3, 4.6, and 4.7 the fundamental homoclinic point has coordinates which are algebraic numbers (mod 1). It is not difficult to see that expansive algebraic $\mathbb{Z}$-actions which are realized as subshifts of $(\mathbb{T}^n)^{\mathbb{Z}}$ as in §2 always have coordinates whose components are algebraic. However, the following example shows that this may fail when $d \geq 2$.

Example 5.8. (*An expansive $\mathbb{Z}^2$-action having transcendental homoclinic points.*) Let $f(u_1,u_2) = 4 - u_1 - u_2 - u_1^{-1}u_2^{-1}$. Then α_{R_2/fR_2} is expansive on X_{R_2/fR_2}. The Fourier transform F of $\widetilde{f}$ is $F(s,t) = 4 - e^{2\pi i s} - e^{2\pi i t} - e^{-2\pi i(s+t)}$. Expansion by geometric series shows that

$$\begin{aligned} w_{\mathbf{0}}^{\triangle} = \left(\frac{1}{F}\right)^{\widehat{}}(\mathbf{0}) &= \frac{1}{4}\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3}\, 4^{-3n} \\ &= \frac{1}{4}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n\, n!} \left(\frac{3}{4}\right)^n = \frac{1}{4}\, H\left(\frac{1}{3}, \frac{2}{3}, 1; \frac{3}{4}\right), \end{aligned}$$

where $(a)_n = a(a+1)\dots(a+n-1)$ and $H(a,b,c;z)$ is the hypergeometric function. This value of the hypergeometric function is known to be transcendental [19]. Thus the $\mathbf{0}$th coordinate of the fundamental homoclinic point for α_{R_2/fR_2} is transcendental.

6. Splitting skew products

Let α be an algebraic $\mathbb{Z}^d$-action on X and suppose that $Y \subset X$ is an α-invariant closed subgroup. By taking a Borel cross-section to the quotient map $X \to X/Y$ we can represent α as a twisted skew product with base action $\alpha_{X/Y}$, and this approach is useful in deriving dynamical properties of α from those of $\alpha_{X/Y}$ and α_Y. More generally, let (Ω, μ) be a probability measure space and β be a measure-preserving $\mathbb{Z}^d$-action on Ω. A measurable function $\psi\colon \Omega \times \mathbb{Z}^d \longrightarrow X$ is a *twisted cocycle* (for α) provided that

$$\psi(\omega, \mathbf{m}+\mathbf{n}) = \alpha^{\mathbf{m}}\psi(\omega, \mathbf{n}) + \psi(\beta^{\mathbf{n}}\omega, \mathbf{m}) \tag{6.1}$$

for all $\omega \in \Omega$ and $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$. The *twisted skew product* $\mathbb{Z}^d$*-action* $\beta \times_\psi \alpha$ on $\Omega \times X$ is given by

$$(\beta \times_\psi \alpha)^{\mathbf{n}}(\omega, x) = (\beta^{\mathbf{n}}\omega, \alpha^{\mathbf{n}}x + \psi(\omega, \mathbf{n})),$$

where (6.1) shows this defines a $\mathbb{Z}^d$-action. The direct product action $\beta \times \alpha$ corresponds to $\psi \equiv 0_X$. Clearly $\beta \times_\psi \alpha$ preserves the product measure $\mu \times \lambda_X$.

For a single automorphism obeying weak specification it was shown in [6] that every twisted skew product is measurably isomorphic to the direct product of the base transformation and the automorphism via a map that translates fibers. One use of this result is a simpler proof of the Bernoullicity of ergodic toral automorphisms that avoids the delicate Diophantine arguments of earlier proofs [6, Thm. 6.3]. Here we extend this splitting result to algebraic $\mathbb{Z}^d$-actions obeying weak specification.

Let $\theta\colon \Omega \to X$ be measurable and define $\Theta\colon \Omega\times X \longrightarrow \Omega\times X$ by $\Theta(\omega, x) = (\omega, x+\theta(\omega))$. For $\mathbf{n} \in \mathbb{Z}^d$ the conjugacy relation $\Theta\cdot(\beta\times_\psi\alpha)^{\mathbf{n}} = (\beta\times\alpha)^{\mathbf{n}}\cdot\Theta$ is equivalent to

$$\psi(\omega, \mathbf{n}) = \alpha^{\mathbf{n}}\theta(\omega) - \theta(\beta^{\mathbf{n}}\omega). \tag{6.2}$$

Hence given α, β, and ψ, we want to solve (6.2) for θ. For technical simplicity we assume β is *aperiodic*, i.e., $\beta^{\mathbf{n}}\omega \neq \omega$ for all $\omega \in \Omega$ and $\mathbf{n} \neq \mathbf{0}$.

Theorem 6.1. *Let α be an algebraic $\mathbb{Z}^d$-action on X satisfying weak specification and β be an aperiodic measure-preserving $\mathbb{Z}^d$-action on (Ω, μ). For every twisted cocycle $\psi\colon \Omega \times \mathbb{Z} \longrightarrow X$ there is a measurable function $\theta\colon \Omega \longrightarrow X$ satisfying (6.2) for all $\omega \in \Omega$ and $\mathbf{n} \in \mathbb{Z}^d$. Hence $\beta \times_\psi \alpha$ is measurably isomorphic to $\beta \times \alpha$ via the map $(\omega, x) \mapsto (\omega, x + \theta(x))$.*

Proof. For $S \subset \mathbb{Z}^d$ and $F \subset \Omega$ it is convenient to let SF denote $\bigcup_{\mathbf{n}\in S}\beta^{\mathbf{n}}F$. Fix a decreasing sequence (ϵ_k) of positive numbers with $\sum_{k=1}^{\infty}\epsilon_k < \infty$. Let $p(\epsilon)$ denote the separation function coming from weak specification for α.

By using Rohlin's lemma for aperiodic $\mathbb{Z}^d$-actions [5], we can find for each $k \geq 1$ a measurable set F_k and a cube $S_k = \{0, 1, \dots, q_k\}^d \subset \mathbb{Z}^d$ satisfying the following properties.

(a) The sets $\beta^{\mathbf{n}}F_k$ are disjoint for $\mathbf{n} \in S_k$ and their union $E_k = S_kF_k$ has $\mu(E_k) > 1 - \epsilon_k$.
(b) $E_1 \subset E_2 \subset E_3 \subset \cdots$.
(c) Almost every $\omega \in \Omega$ has the property that for every $n \geq 1$ there is a $k \geq 1$ such that $\{-n, \dots, n\}^d\,\omega \subset E_k$.

(d) By (c), if $\omega \in F_{k+1}$ then $(S_{k+1}\omega) \cap E_k = \bigcup_{j=1}^{r}(S_k + \mathbf{n}_j)\,\omega$, and we require that the cubes $S_k + \mathbf{n}_1, \dots, S_k + \mathbf{n}_r$ be separated by at least $p(\epsilon_k)$.

We inductively construct measurable $\theta_k\colon E_k \longrightarrow X$ such that θ_{k+1} is ϵ_k-uniformly close to θ_k on E_k and such that θ_k satisfies $\psi(\omega, \mathbf{n}) = \alpha^{\mathbf{n}}\theta_k(\omega) - \theta(\beta^{\mathbf{n}}\omega)$ for all $\omega \in F_k$ and $\mathbf{n} \in S_k$. It follows from property (a) that $\theta = \lim_{k\to\infty} \theta_k$ is defined almost everywhere and is measurable, and from (c) that θ satisfies (6.2) for a set of full measure in Ω and all $\mathbf{n} \in \mathbb{Z}^d$.

Start by defining θ_1 measurably but otherwise arbitrarily on F_1, and extend θ_1 to E_1 by $\theta_1(\beta^{\mathbf{n}}\omega) = \alpha^{\mathbf{n}}\theta_1(\omega) - \psi(\omega, \mathbf{n})$ for $\omega \in F_1$ and $\mathbf{n} \in S_1$. Assume inductively that θ_k is defined on E_k such that $\theta_k(\beta^{\mathbf{n}}\omega) = \alpha^{\mathbf{n}}\theta_k(\omega) - \psi(\omega, \mathbf{n})$ for $\omega \in F_k$ and $\mathbf{n} \in S_k$. Let $\widetilde{\theta}_{k+1}$ be an arbitrary measurable function on F_{k+1}, and extend it to E_{k+1} as before. Fix $\omega_0 \in F_{k+1}$. Then

$$(S_{k+1}\omega_0) \cap E_k = \bigcup_{j=1}^{r} (S_k + \mathbf{n}_j)\,\omega_0.$$

If ω_j denotes $\beta^{\mathbf{n}_j}\omega_0$, then for all $\mathbf{m} \in S_k$ we have that

$$\begin{aligned}
&\widetilde{\theta}_{k+1}(\beta^{\mathbf{m}}\omega_j) - \theta_k(\beta^{\mathbf{m}}\omega_j)\\
&\qquad = \alpha^{\mathbf{m}}\widetilde{\theta}_{k+1}(\omega_j) - \psi(\omega_j, \mathbf{m}) - [\alpha^{\mathbf{m}}\theta_k(\omega_j) - \psi(\omega_j, \mathbf{m})]\\
&\qquad = \alpha^{\mathbf{m}}[\widetilde{\theta}_{k+1}(\omega_j) - \theta_k(\omega_j)].
\end{aligned}$$

Let $x_j = \widetilde{\theta}_{k+1}(\omega_j) - \theta_k(\omega_j)$. Since the cubes $S_k + \mathbf{n}_1, \dots, S_k + \mathbf{n}_r$ are separated by at least $p(\epsilon_k)$, weak specification for α shows that there is an $x \in X$ such that $\rho(\alpha^{\mathbf{m}+\mathbf{n}_j}x, \alpha^{\mathbf{m}}x_j) < \epsilon_k$ for $\mathbf{m} \in S_k$ and $1 \le j \le r$. By defining $\theta_{k+1}(\omega_0) = \widetilde{\theta}_{k+1}(\omega_0) - x$ and extending to $S_{k+1}\omega_0$ as before, we obtain that θ_{k+1} is ϵ_k-uniformly close to θ_k on $(S_{k+1}\omega_0) \cap E_k$.

A word is needed about measurability of θ_{k+1} in this construction. For each $\omega_0 \in F_{k+1}$ we adjust $\widetilde{\theta}_{k+1}(\omega_0)$ by an amount x determined from weak specification. By using $\le$ inequalities in the specification definition, we obtain for each $\omega_0 \in F_{k+1}$ a nonempty compact set $K(\omega_0)$ of allowable adjustments, where $K(\omega_0)$ varies measurably with ω_0 by its definition. Standard selection theorems then show that there is a measurable $\kappa\colon F_{k+1} \longrightarrow X$ such that $\kappa(\omega_0) \in K(\omega_0)$. Then $\theta_{k+1} = \widetilde{\theta}_{k+1} - \kappa$ is measurable. □

Example 6.2. (*Two expansive algebraic $\mathbb{Z}^d$-actions that are measurably but not algebraically or topologically isomorphic.*) Let f be a nonunit polynomial in $R = R_d$ such that $V_{\mathbb{C}}(f) \cap \mathbb{S}^d \neq \varnothing$. The natural quotient map $R/f^2R \longrightarrow R/fR$ dualizes to show that $Y = X_{R/fR}$ is an α_{R/f^2R}-invariant subgroup of $X = X_{R/f^2R}$. As pointed out at the start of this section, we can therefore regard α_{R/f^2R} as a twisted skew product $\beta \times_\psi \alpha$, where the base action is $\beta = \alpha_{fR/f^2R} \cong \alpha_{R/fR}$ and the fiber action is $\alpha = \alpha_{R/fR}$. Since α is expansive and has completely positive entropy, it satisfies weak specification by Theorem 5.2. Hence by Theorem 6.1 we see that α_{R/f^2R} is measurably isomorphic to $\alpha_{R/fR} \times \alpha_{R/fR}$. The existence of an isomorphism between these actions (although not of the precise form given by Theorem 6.1) also follows from the deeper facts that they are both Bernoulli $\mathbb{Z}^d$-actions [15] with the same entropy [8].

However, there is no group isomorphism $\phi\colon X_{R/f^2R} \longrightarrow X_{R/fR} \times X_{R/fR}$ intertwining α_{R/f^2R} with $\alpha_{R/fR} \times \alpha_{R/fR}$. For $\widehat{\phi}$ would give an R-module isomorphism of R/f^2R with $(R/fR) \times (R/fR)$, which is clearly impossible since f annihilates the second R-module but not the first. Since $X_{R/fR}$ is connected, Theorem 5.9 of [15] shows that these actions are not even topologically conjugate.

7. Homoclinic groups of nonexpansive algebraic $\mathbb{Z}^d$-actions

The purpose of this section is to show by a series of nonexpansive examples that most of our results for expansive actions do not extend to arbitrary actions.

We first describe a basic reason why nonexpansive actions can have more complicated homoclinic groups. Let $f \in R_d$ and consider the $\mathbb{Z}^d$-action $\alpha = \alpha_{R_d/fR_d}$. As in §2 we identify $X = X_{R_d/fR_d}$ as a shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}^d}$. Let $\eta\colon \ell^\infty(\mathbb{Z}^d, \mathbb{R}) \longrightarrow \mathbb{T}^{\mathbb{Z}^d}$ be defined as before by reducing each coordinate (mod 1). Recall the definition of $\widetilde{f} \in c_0(\mathbb{Z}^d, \mathbb{R})$ from the proof of Lemma 4.5. Say that $w \in c_0(\mathbb{Z}^d, \mathbb{R})$ is a *linear homoclinic point* for α if $\widetilde{f} * w = 0$. If $w \neq 0$ is such a point, then for every $t \in \mathbb{R}$ we see that $\eta(tw) \in \Delta_\alpha(X)$, and hence $\Delta_\alpha(X)$ is uncountable. But Lemma 3.2 shows that if α is expansive this cannot happen. We have repeatedly used this basic property that expansive actions never have nonzero linear homoclinic points in previous sections.

In contrast, for nonexpansive examples the set $V_{\mathbb{C}}(f) \cap \mathbb{S}^d$ may be sufficiently large to support measures whose Fourier transform decays to 0 at infinity, and this provides a rich supply of linear homoclinic points. Examples 7.3 and 7.5 employ this idea.

Example 7.1. (*Completely positive entropy and trivial homoclinic group.*) Let $g(u) \in \mathbb{Z}[u]$ be irreducible and monic, have constant term ± 1, and have some but not all of its roots on $\mathbb{S}$. Set $V_{\mathbb{C}}(g) \cap \mathbb{S} = \{\xi_1, \dots, \xi_r\}$. If we define $f \in R_2$ by $f(u_1, u_2) = g(u_1)$, then $V_{\mathbb{C}}(f) = \bigcup_{j=1}^r \{\xi_j\} \times \mathbb{S}$. The $\mathbb{Z}^2$-action $\alpha = \alpha_{R_2/fR_2}$ on $X = X_{R_2/fR_2}$ has completely positive entropy by Lemma 2.1(4).

To prove that $\Delta_\alpha(X) = \{0_X\}$, we first realize α and X in the following way. Let $k = \deg g$, let A be the companion matrix of g, and let ϕ be the automorphism of $\mathbb{T}^k$ induced by A. Then $X \cong (\mathbb{T}^k)^{\mathbb{Z}}$, and $\alpha^{(1,0)}$ acts on X by applying ϕ to each coordinate, while $\alpha^{(0,1)}$ acts on X as the shift. Then every $x = (x_n) \in \Delta_\alpha(X)$ must have each $x_n \in \Delta_\phi(\mathbb{T}^k)$. But Example 3.4 shows that $\Delta_\phi(\mathbb{T}^k) = \{0_{\mathbb{T}^k}\}$. Hence $\Delta_\alpha(X) = \{0_X\}$.

Example 7.2. (*Completely positive entropy and countably infinite homoclinic group.*) Let $f(u_1, u_2) = 2 - u_1 - u_2 \in R_2$. Then $V_{\mathbb{C}}(f) \cap \mathbb{S}^2 = \{(1, 1)\}$, and so $\alpha = \alpha_{R_2/fR_2}$ is not expansive on $X = X_{R_d/fR_2}$. The Fourier transform F of $\widetilde{f}$ is given by $F(s,t) = 2 - e^{2\pi i s} - e^{2\pi i t}$, and although $1/F$ is unbounded on $\mathbb{T}^2$ one can easily verify that $1/F \in L^1(\mathbb{T}^2, \lambda_{\mathbb{T}^2})$. The Fourier series of $1/F$ is computed exactly as in Example 4.6, resulting in the point $w^\triangle \in \ell^\infty(\mathbb{Z}^2, \mathbb{R})$ defined by

$$w^\triangle_{(-m,-n)} = \begin{cases} \dfrac{1}{2^{m+n+1}} \dbinom{m+n}{n} & \text{if } m \geq 0 \text{ and } n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The Riemann-Lebesgue Lemma shows that $w^\triangle \in c_0(\mathbb{Z}^2, \mathbb{R})$, which can also be deduced from the explicit formula above since the nonzero terms are binomial probabilities. Hence $x^\triangle = \eta(w^\triangle) \in \Delta_\alpha(X)$, and the same arguments as in the proof of Lemma 4.5 show that the map $R_2/fR_2 \longrightarrow \Delta_\alpha(X)$ given by $g + fR_2 \mapsto g(\sigma)x^\triangle$ is an isomorphism.

Are there other points in $\Delta_\alpha(X)$? Suppose that $x \in \Delta_\alpha(X)$ and choose $v \in \ell^\infty(\mathbb{Z}^2, \mathbb{R})$ with $\|v\|_\infty \leq 1/2$ and $\eta(v) = x$. Then $\widetilde{f} * v \in \ell^\infty(\mathbb{Z}^2, \mathbb{Z}) \cap c_0(\mathbb{Z}^2, \mathbb{R})$, so that there exists $g \in R_2$ with $\widetilde{f} * v = \widetilde{g}$. Since $\widetilde{g} = \widetilde{g} * \widetilde{f} * w^\triangle$, we see that $w = v - \widetilde{g} * w^\triangle \in c_0(\mathbb{Z}^2, \mathbb{R})$ and $\widetilde{f} * w = 0$. If α were expansive, this would be enough to conclude that $w = 0$ as above, and hence that $x = \eta(v) = g(\sigma)x^\triangle$, which would prove that there are no additional homoclinic points. This line of reasoning is correct, but requires a different argument to show that if $w \in c_0(\mathbb{Z}^2, \mathbb{R})$ and $\widetilde{f} * w = 0$, then $w = 0$.

Start by adjusting w by a translation so that $|w_{(0,0)}| = \sup_{\mathbf{n} \in \mathbb{Z}^2} |w_{\mathbf{n}}|$. Since $w \in c_0(\mathbb{Z}^2, \mathbb{R})$, there is an $m > 0$ such that $|w_{(k,m-k)}| \leq \frac{1}{2}|w_{(0,0)}|$ for all $0 \leq k \leq m$. Since $\widetilde{f} * w$ we see that

$$\begin{aligned} w_{(0,0)} &= \frac{1}{2} w_{(1,0)} + \frac{1}{2} w_{(0,1)} = \frac{1}{4} w_{(2,0)} + \frac{1}{2} w_{(1,1)} + \frac{1}{4} w_{(0,2)} = \dots \\ &= \sum_{k=0}^{m} \frac{1}{2^m} \binom{m}{k} w_{(k,m-k)}. \end{aligned}$$

Hence

$$|w_{(0,0)}| \leq \sum_{k=0}^{m} \frac{1}{2^m} \binom{m}{k} |w_{(k,m-k)}| \leq \frac{1}{2} |w_{(0,0)}|.$$

This proves that $w_{(0,0)} = 0$, and so $w = 0$. We have therefore shown that all homoclinic points have the form $g(\sigma)x^\triangle$, so that $\Delta_\alpha(X)$ is countable. The proof of Lemma 4.5 applies to establish that $\Delta_\alpha(X)$ is dense in X.

There is a simple idea behind the proof in the previous paragraph. The condition $\widetilde{f} * w = 0$ means that w is "harmonic" in the sense that $w_{(m,n)} = \frac{1}{2}[w_{(m+1,n)} + w_{(m,n+1)}]$. Now harmonic points must satisfy an analogue of the Maximum Principle, namely that they attain their maximum value over a "region" on its "boundary." But no point in $c_0(\mathbb{Z}^2, \mathbb{R})$ can satisfy this principle unless it is zero. The arguments in this example therefore apply to all $f \in R_d$ having exactly one positive coefficient such that $\sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n}) = 0$.

Observe that here the diagonal coordinates of w are

$$w_{(-n,-n)} = \frac{1}{2^{2n+1}}\binom{2n}{n} \sim \frac{1}{2\sqrt{\pi n}}$$

(see [2, p. 75]), which decay slowly as $n \to \infty$. Thus the homoclinic point $x^{\triangle}$ does not exhibit the exponential decay that by Lemma 4.3 must occur for homoclinic points of expansive actions. In particular, homoclinic points with slow decay seem useless in determining whether an action obeys specification. For example, we do not know whether the action of this example satisfies any of the specifications defined in §5.

Example 7.3. (*Completely positive entropy and uncountable homoclinic group.*) For this example let $f(u_1, u_2) = 3 - u_1 - u_1^{-1} - u_2 - u_2^{-1}$. Then the Fourier transform F of $\widetilde{f}$ is given by $F(s,t) = 3 - 2\cos 2\pi s - 2\cos 2\pi t$. Let $K = \{(s,t) \in \mathbb{T}^2 : F(s,t) = 0\}$. This is a smooth real-analytic curve in $\mathbb{T}^2$, viz.

$$t = \pm\frac{1}{2\pi}\cos^{-1}\left(\frac{3}{2} - \cos 2\pi s\right), \qquad -\frac{1}{6} \le s \le \frac{1}{6}.$$

Then $V_{\mathbb{C}}(f) \cap \mathbb{S}^2 = \{(e^{2\pi i s}, e^{2\pi i t}) : (s,t) \in K\}$, so that $\alpha = \alpha_{R_2/fR_2}$ is not expansive on $X = X_{R_2/fR_2}$. In contrast to the previous example, here $1/F \notin L^1(\mathbb{T}^2, \lambda_{\mathbb{T}^2})$ so that the Fourier methods of Lemma 4.5 fail. The existence of linear homoclinic points accounts for the uncountability of $\Delta_\alpha(X)$.

By identifying $\mathbb{T}^2$ with the subset $[-1/2, 1/2)^2 \subset \mathbb{R}^2$, we may consider K as a smooth curve in $\mathbb{R}^2$. Let ν_K denote the measure on K induced by arc length, and $\psi \in C^\infty(K)$ be any smooth non-zero function on K. Define a measure μ on K by $d\mu = \psi\, d\nu_K$. Since μ is supported on K and F vanishes there, we see that

$$(\widetilde{f} * \widehat{\mu})\widehat{\ } = (\widetilde{f})\widehat{\ }\cdot \mu = F \cdot \mu = 0.$$

Thus $\widehat{\mu} \in \ell^\infty(\mathbb{Z}^2, \mathbb{C})$ satisfies $\widetilde{f} * \widehat{\mu} = 0$. Now K has curvature bounded away from zero, so it follows from [18, Thm. 1 of §VIII.3.1] that there is a constant $C > 0$ such that

$$|\widehat{\mu}(\mathbf{n})| \le C\|\mathbf{n}\|^{-1/2} \quad \text{for all } \mathbf{n} \in \mathbb{Z}^2. \tag{7.1}$$

Hence $v = \operatorname{Re}\widehat{\mu}$, $w = \operatorname{Im}\widehat{\mu} \in \ell^\infty(\mathbb{Z}^2, \mathbb{R})$ are linear homoclinic points that are not both zero, say $v \ne 0$. Then $\eta(tv) \in \Delta_\alpha(X)$ for all $t \in \mathbb{R}$, so that $\Delta_\alpha(X)$ is uncountable. Indeed, distinct $\psi \in C^\infty(K)$ yield distinct $\widehat{\mu}$, so that $\Delta_\alpha(X)$ encompasses the complexity of $C^\infty(K)$.

What is essential for the decay estimate (7.1) is that the curve K not have infinite-order contact with any line, which is guaranteed here by the curvature of K being bounded away from zero. This curvature is missing in Example 7.1, and explains why that example has no linear homoclinic points.

This example was suggested to us by Hart Smith as a finite-difference analogue of the partial differential operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 - 1$. This operator has nonzero solutions that do not obey the Maximum Principle and which decay to 0 at infinity, precisely the behavior needed to find linear homoclinic points in the finite-difference setting.

Example 7.4. (*Zero entropy and trivial homoclinic group.*) Choose $g(u)$, $h(u) \in \mathbb{Z}[u]$ to be monic and irreducible, have constant term ± 1, and have some but not all of their roots on $\mathbb{S}$. Set $V_{\mathbb{C}}(g) \cap \mathbb{S} = \{\xi_1, \dots, \xi_r\}$ and $V_{\mathbb{C}}(h) = \{\eta_1, \dots, \eta_s\}$. We may assume that $\xi_i^m \eta_j^n \neq 1$ for $1 \le i \le r$, $1 \le j \le s$, and $(m,n) \neq (0,0)$. Let $k = \deg g$, $\ell = \deg h$, and A, B be the companion matrices of g, h, respectively. Denote the $\ell \times \ell$ identity matrix by I_ℓ. Define ϕ to be the automorphism of $\mathbb{T}^{k\ell}$ induced by $A \otimes I_\ell$ and ψ to be that induced by $I_k \otimes B$.

If $\mathfrak{a}$ is the ideal in R_2 generated by $g(u_1)$ and $h(u_2)$, then $X = X_{R_2/\mathfrak{a}} \cong \mathbb{T}^{k\ell}$ and the $\mathbb{Z}^2$-action $\alpha = \alpha_{R_2/\mathfrak{a}}$ is generated by ϕ and ψ. Our assumptions on the ξ_i and η_j show that α mixing. Now $h(\alpha) = 0$ since smooth $\mathbb{Z}^2$-actions have topological entropy zero. Note that ϕ is isomorphic to the direct product of ℓ copies of an ergodic nonhyperbolic toral automorphism, and so $\Delta_\phi(X) = \{0_X\}$ by Example 3.4. Hence $\Delta_\alpha(X) \subset \Delta_\phi(X)$ is also trivial.

Example 7.5. (*Zero entropy and uncountable homoclinic group.*) Define $f, g \in R_3$ by

$$f(u_1,u_2,u_3) = 5 - u_1 - u_1^{-1} - u_2 - u_2^{-1} - u_3 - u_3^{-1},$$
$$g(u_1,u_2,u_3) = 3 - u_1 - u_1^{-1} - u_1u_3 - u_1^{-1}u_3^{-1},$$

and put $\mathfrak{a} = \langle f, g \rangle$. Since f and g have no common factor in R_3, it follows that every prime ideal associated to $R_3/\mathfrak{a}$ is nonprincipal, and hence $h(\alpha_{R_3/\mathfrak{a}}) = 0$ by Lemma 2.1(3).

In order to describe $V_{\mathbb{C}}(\mathfrak{a})$, let $\tau\colon \mathbb{T}^3 \longrightarrow \mathbb{S}^3$ be the isomorphism given by

$$\tau(s,t,u) = (e^{2\pi i s}, e^{2\pi i t}, e^{2\pi i u}).$$

Then

$$\begin{aligned} V_1 &= \tau^{-1}[V_{\mathbb{C}}(f) \cap \mathbb{S}^3] \\ &= \{\, (s,t,u) \in \mathbb{T}^3 : 5 - 2\cos 2\pi s - 2\cos 2\pi t - 2\cos 2\pi u = 0 \,\} \end{aligned}$$

and

$$\begin{aligned} V_2 &= \tau^{-1}[V_{\mathbb{C}}(g) \cap \mathbb{S}^3] \\ &= \{\, (s,t,u) \in \mathbb{T}^3 : 3 - 2\cos 2\pi s - 2\cos 2\pi(s+u) = 0 \,\}. \end{aligned}$$

Here V_1 is a 2-dimensional spheroid in $\mathbb{T}^3$ while V_2 is a skewed cylinder piercing through V_1. Their intersection $V = V_1 \cap V_2$ is the disjoint union of two real-analytic curves K_1 and K_2. We give a parametric representation of these curves in terms of s as follows.

First observe that $(s,t,u) \in V_2$ if and only if

$$u = \frac{1}{2\pi}\cos^{-1}\left(\frac{3}{2} - \cos 2\pi s\right) - s, \tag{7.2}$$

which determines u in terms of s on V. Since $f - g = 0$ on V, we obtain that

$$2 - 2\cos 2\pi t - 2\cos 2\pi u + 2\cos 2\pi(s+u) = 0.$$

Cancelling the factor 2 and noting that $\cos 2\pi(s+u) = \frac{3}{2} - \cos 2\pi s$ on V_2, we see that on V

$$\cos 2\pi t = \frac{5}{2} - \cos 2\pi u - \cos 2\pi s.$$

Hence by (7.2)

$$t = \frac{1}{2\pi}\cos^{-1}\left[\frac{5}{2} - \cos 2\pi s - \cos\left\{\cos^{-1}\left(\frac{3}{2} - \cos 2\pi s\right) - 2\pi s\right\}\right]. \tag{7.3}$$

Taking into account the appropriate branches of $\cos^{-1}$, the equations (7.2) and (7.3) describe the two curves K_1 and K_2 in $\mathbb{T}^3$.

Identify $\mathbb{T}^3$ with $[-1/2, 1/2)^3 \subset \mathbb{R}^3$, and so consider $V = K_1 \cup K_2$ as a subset of $\mathbb{R}^3$. It can be verified from our parametric representation that the curves K_i are real-analytic and neither is contained in a hyperplane. More specifically, each K_i has contact of order at most 2 with each 2-dimensional hyperplane, so that, in the terminology of [18], $V = K_1 \cup K_2$ has type 2. Let ν_V be the measure on V induced by arc length, $\psi \in C^\infty(V)$, and μ be the measure defined by $d\mu = \psi\, d\nu_V$. By [18, Thm. 2 of §VIII.3.2], it follows that there is a constant $C > 0$ such that

$$|\widehat{\mu}(\mathbf{n})| \le C\|\mathbf{n}\|^{-1/2} \quad \text{for all } \mathbf{n} \in \mathbb{Z}^3.$$

As in Example 7.3, this provides a linear homoclinic point in $c_0(\mathbb{Z}^3, \mathbb{R})$, and so by scaling gives an uncountable number of homoclinic points for α.

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