

1. (15 points) The graphs to the right are parabolas, with the following formulas:

$$f(x) = x^2 - 5x + 15$$

$$g(x) = -x^2 + 6x + 10$$

- a) Find the values of x at which the two graphs cross.

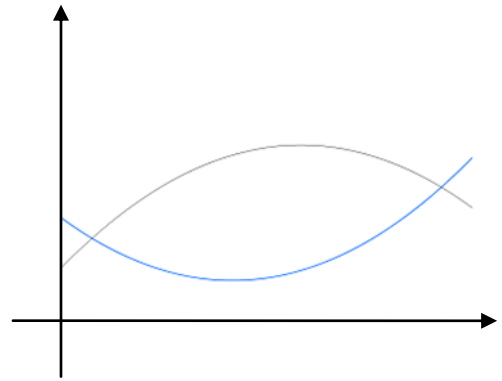
The two graphs cross when $f(x) = g(x)$, i.e. when

$$x^2 - 5x + 15 = -x^2 + 6x + 10$$

This is a quadratic equation, so we solve it via the quadratic formula:

$$2x^2 - 11x + 5 = 0$$

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4 \cdot 2 \cdot 5}}{2 \cdot 2} = \frac{11 \pm \sqrt{81}}{4} = \frac{11 \pm 9}{4} = 0.5 \text{ and } 5$$



ANSWER: at $x = \underline{0.5}$ and at $x = \underline{5}$

- b) Find the longest interval over which both functions are increasing. Fully justify your answer.

The graph of $f(x) = x^2 - 5x + 15$ is a concave up parabola, so it increases from its vertex $x = \frac{-(-5)}{2} = 2.5$ on.

The graph of $g(x) = -x^2 + 6x + 10$ is a concave down parabola, so it increases before its vertex $x = \frac{-6}{2(-1)} = 3$.

So the longest interval over which both functions are increasing is from $x=2.5$ to $x=3$.

ANSWER: From $x = \underline{2.5}$ to $x = \underline{3}$

- c) Find the value of x at which $\frac{f(x)-f(0)}{x} = 7$

Rewrite the given expression $\frac{f(x)-f(0)}{x} = 7$ in terms of the given function $f(x) = x^2 - 5x + 15$, then simplify and solve for x :

$$\frac{f(x) - f(0)}{x} = 7$$

$$\frac{x^2 - 5x + 15 - 15}{x} = 7$$

$$\frac{x^2 - 5x}{x} = 7$$

$$\frac{x(x - 5)}{x} = 7$$

$$x - 5 = 7$$

$$x = 12$$

ANSWER: at $x = \underline{12}$

2. (20 pts) You produce and sell Cans of Cat Food, in order sizes of 1 to 150 Cans.

Each Can costs \$1.50 to produce. Your fixed costs are \$30.

To encourage larger orders, you offer a volume discount as follows: the price for an order of **one** Can is \$2.99, and you **decrease** the price per Can **by 1 cent** for each additional Can ordered. For instance, if a customer buys 5 Cans, the selling price is \$2.95 per Can.

a) (3 pts) Write down a linear formula, in terms of quantity q ordered, for the selling price per Can.

A linear formula in terms of q is of the form: $p(q)=mq+b$. To compute the slope:

- we can either notice that the price drops by 1 cent = \$0.01 for each extra item ordered, so $m = -0.01$
- or, we can compute the slope via the two points provided: $p(1)=2.99$, $p(5)=2.95$, so

$$m = \frac{2.95 - 2.99}{5 - 1} = -\frac{0.04}{4} = -0.01$$

To compute the y-intercept, substitute one of the given points in the equation so far: $p(q) = -0.01q + b$.

For instance, when $q = 1, p = 2.99$, so $2.99 = -0.01 \cdot 1 + b$, which gives $b = 2.99 + 0.01 = 3$

$$p(q) = \underline{-0.01q + 3}$$

b) (4 pts) Write down formulas in terms of q for the Total Revenue and the Total Cost for an order of q Cans.

Recall that $TR(q) = p(q) \cdot q$. Using part (a) we get: $TR(q) = (-0.01q + 3) \cdot q = -0.01q^2 + 3q$.

Also, if the cost is constant at \$1.50 per can, then $VC(q) = 1.50 \cdot q$, and $TC(q) = VC(q) + FC = 1.50q + 30$.

(In particular, remember NOT to mix costs and revenue!)

$$TR(q) = \underline{-0.01q^2 + 3q}$$

$$TC(q) = \underline{1.50q + 30}$$

Note: To answer the following questions you need the TR function from part (b). If you could not answer part (b), circle and use the formula $TR(q) = 5q - 0.02q^2$ instead (this is not the correct answer in part b!)

c) (5 pts) Compute the formula in terms of q for the Marginal Revenue. Show all steps & simplify your answer.

$$\begin{aligned} MR(q) &= TR(q+1) - TR(q) \\ &= [-0.01(q+1)^2 + 3(q+1)] - [-0.01q^2 + 3q] \\ &= -0.01(q^2 + 2q + 1) + 3(q+1) + 0.01q^2 - 3q \\ &= -0.01q^2 - 0.02q - 0.01 + 3q + 3 + 0.01q^2 - 3q \\ &= -0.02q + 2.99 \end{aligned}$$

$$\text{ANSWER: } MR(q) = -0.02q + 2.99$$

[If using alternative: $TR(q) = 5q - 0.02q^2$, then you should get $MR(q) = -0.04q + 4.98$]

d) (6 points) Compute the largest profit possible.

Method I: Compute a formula for the profit:

$$Profit(q) = TR(q) - TC(q) = -0.01q^2 + 3q - (1.50q + 30) = -0.01q^2 + 1.50q - 30$$

Since the graph of the profit function is maximal at its vertex $q = \frac{-1.50}{2(-0.01)} = 75$ cans, the max profit would be

$$Profit(75) = -0.01(75)^2 + 1.50(75) - 30 = 26.25$$

ANSWER: Max profit is 26.25 dollars

[If using alternative: $TR(q) = 5q - 0.02q^2$, then you should get $Profit(q) = -0.02q^2 + 3.50q - 30$, with vertex at $q=87.5$, which rounded to nearest whole number gives $q=87$ or $q=88$. Max profit is profit at either $q=87$ or $q=88$, namely \$123.12.]

Method II: Compute the quantity where $MR(q)=MC(q)$ (noting that $MC(q)=1.50$, because each item costs 1.50), then round to the next whole number quantity and compute the corresponding profit:

$$MR = MC: -0.02q + 2.99 = 1.50, \text{ which gives } q = \frac{1.50-2.99}{-0.02} = 74.5. \text{ Round to } q=75.$$

$$Profit(75) = -0.01(75)^2 + 1.50(75) - 30 = 26.25$$

ANSWER: Max profit is _____26.25_____ dollars

[If using alternative: $MR(q) = -0.04q + 4.98 = MC(q) = 1.50$, then you get $q=87$. Profit at $q=87$ is \$123.12.]

3. (15 pts) The marginal revenue and marginal cost at q Items are given by the following linear functions:

$$MR(q) = -0.5q + b \quad \text{dollars}$$

$$MC(q) = 0.4q + 7 \quad \text{dollars}$$

In addition, the average cost (in dollars per Item) is given by the function:

$$AC(q) = 0.2q + 6.8 + \frac{38}{q}$$

- a) What is the change in total cost if q increases from 4 to 5 Items?

$$MC(4) = 0.4 \cdot 4 + 7 = 8.6$$

ANSWER: _____8.6_____ dollars

- b) Recall that $MR(q) = -0.5q + b$. Compute a value of b that results in the profit being maximal at $q = 10$ Items.

$$\text{Set } MR(10)=MC(10) \text{ and solve for } b: -0.5 \cdot 10 + b = 0.4 \cdot 10 + 7$$

$$-5 + b = 11$$

$$b = 16$$

ANSWER: $b =$ _____16_____

- c) Compute the breakeven price.

Here AC is not a quadratic function, so we cannot use the vertex formula to help determine the lowest value of AC ! We must use the other method: **y-coordinate of point where $MC=AC$** :

$$0.4q + 7 = 0.2q + 6.8 + \frac{38}{q}$$

Multiply both sides of the equation by q to cancel the denominator:

$$q \cdot (0.4q + 7) = q \cdot \left(0.2q + 6.8 + \frac{38}{q}\right)$$

$$0.4q^2 + 7q = 0.2q^2 + 6.8q + 38$$

$$0.2q^2 + 0.2q - 38 = 0$$

Applying the quadratic formula: $q = -14.2$, or $q = 13.293 \dots$ We cannot have negative quantities, so $q = 13.293 \dots$

To get the y-coordinate of the intersection point, we can compute $MC(13.293 \dots) = 0.4(13.293 \dots) + 7 \approx \12.32

ANSWER: BEP = _____12.32_____ dollars per item.