Math 111B,C - Winter 2003
Mid-Term Exam Number Two
Solutions
February 25, 2003

1. We are making and selling surfboard wax in 300 gram bars. To encourage large orders, we give quantity discounts by determining price per bar to be a linear function of the quantity ordered. If an order is for one bar, we charge \$8. For an order of 100 bars, we charge \$5 per bar.

   (a) (5 points) Write a formula that gives the price p per bar for an order of q bars. Keep your answer in fraction form - don't use decimal approximations.
   Solution: We're looking for a function
   $$p(q) = mq + b$$
   using the points $(p, q) = (1, 8)$ and $(p, q) = (100, 5)$. The slope is
   $$m = \frac{8-5}{1-100} = \frac{3}{-99} = -\frac{1}{33}.$$
   So
   $$8 = -\frac{1}{33}(1) + b$$
   $$b = 8 + \frac{1}{33} = \frac{265}{33}.$$
   So the function is
   $$p(q) = -\frac{1}{33}q + \frac{265}{33}.$$

   (b) (5 points) What is the revenue from an order of q bars?
   Solution: Since
   $$TR = p \times q,$$
   we have
   $$TR = q\left(-\frac{1}{33}q + \frac{265}{33}\right).$$

   (c) (5 points) Find the vertex of your revenue function.
   Solution:
   $$TR = -\frac{1}{33}q^2 + \frac{265}{33}q$$
   so the vertex is at $q = \frac{-\frac{265}{33}}{-2\left(\frac{1}{33}\right)} = \frac{265}{2}$. The revenue at $q = \frac{265}{2}$ is
   $$-\frac{1}{33}\left(\frac{265}{2}\right)^2 + \frac{265}{33}\left(\frac{265}{2}\right) = \frac{70225}{132}.$$
   So the vertex is $\left(\frac{265}{2}, \frac{70225}{132}\right)$.

   (d) (2 points) What value of q results in the maximum revenue?
   Solution: From the vertex: $\frac{265}{2}$.

   (e) (2 points) What is the maximum possible revenue?
   Solution: From the vertex: $\frac{70225}{132} = \$532.0075757....$

2. The average cost (AC) of manufacturing an order of q titanium opera glass cases is

$$AC(q) = 4.53 + \frac{5000}{q}$$

while the revenue from selling q cases

$$TR(q) = 200q - q^2.$$

(a) Write out a formula for total cost (TC) as a function of q. Write it in the form $TC = aq + b$.

Solution: Since $AC = \frac{TC}{q}$, we know $TC = AC \times q$. So

$$TC = q(4.53 + \frac{5000}{q}) = 4.53q + 5000.$$

(b) What is the fixed cost (FC) ?

Solution:

$$FC = TC(0) = 4.53(0) + 5000 = \$5000.$$

(c) What is the marginal revenue at $q = 90$?

Solution:

$$MR(q) = TR(q+1) - TR(q) = 200(q+1) - (q+1)^2 - (200q - q^2) =$$

$$200q + 200 - q^2 - 2q - 1 - 200q + q^2 = 200 - 2q - 1 = 199 - 2q$$

so at $q = 90$, MR is $199 - (2)(90) = 19$.

(d) For what q is $MR = MC$?

Solution: Marginal Cost is \$4.53, so we set $MR = 4.53$ and solve for q.

$$199 - 2q = 4.53$$

$$q = 97.235.$$

3. You are in the business of manufacturing electric dog polishers. If you make and sell q **hundred** polishers this year, your revenue in **thousands** of dollars will be

$$TR = -15q^2 + 120q$$

while your costs in **thousands** of dollars will be

$$TC = 2q^2 + 4q + 38$$

Below, give the exact answer from your calculations - don't worry about what a fraction of a polisher means.

(a) What is the smallest number of polishers you can make if you want to break even (i.e., have zero profit)?

Solution: Zero profit means that $TR = TC$. Setting $TR = TC$ and solving, we have

$$-15q^2 + 120q = 2q^2 + 4q + 38$$

$$0 = 17q^2 - 116q + 38$$

so

$$q = \frac{116 \pm \sqrt{116^2 - (4)(17)(38)}}{2(17)}$$

so $q = 0.345032847...$ or $q = 6.47849656....$ Since we want the smallest number of polishers, the answer is $100 \cdot 0.345032847... = 34.5032847...$ polishers.

(b) How many polishers should you make if you want to make the most profit?

Solution: We are interested in the horizontal coordinate of the vertex of the profit function, $TR - TC = -17q^2 + 116q - 38$. That is,

$$q = \frac{-116}{2(-17)} = \frac{58}{17} = 3.4117647...$$

i.e., 341.17647... polishers will maximize the profit.

4. For the first 20 years of a eucalyptus tree's life, its growth rate is given by $4 - 0.2t$ feet per year, where t is the number of years since it was planted.

(a) What is the tree's average rate of growth for the first 10 years of its life?

Solution: Since the growth rate is a linear function of time, we can can the average growth rate by averaging the rate at the beginning and at the end of the time interval. At $t = 0$, the growth rate is 4 while at $t = 10$, the growth rate is 2. So the average growth rate is $\frac{4+2}{2} = 3$.

(b) Assume the tree's height was zero when it was planted. How tall is the tree after 15 years?

Solution: The growth rate after 15 years is $4 - 0.2(15) = 1$. Similar to part (a), the average growth rate over the 15 years is the average of the growth rates at the beginning and at the end:

$$\frac{4+1}{2} = 2.5.$$

Since this is the average growth rate over the 15 years, the tree grew $15 \times 2.5 = 37.5$ feet, and is 37.5 feet tall after 15 years.