VI

1. (16 points)

(a) (4 pts) Bilbo invests $5000 into an account that has a 6.2% annual rate, compounded continuously. How much money is in Bilbo's account in 4 years?

$A(t) = Pe^{rt}$

$A(t) = 5000e^{0.062t}$

$A(4) = 5000e^{0.062 \times 4} \approx 6407.29961$

$\boxed{6407.30}$ dollars

(b) (4 pts) How much must you deposit in an account paying 5% annually, compounded quarterly, to have $920 in 18 months?

$A(t) = P(1+\frac{r}{n})^{nt}$ 18 months $= \frac{18}{12}$ years $= 1.5$ years

$A(t) = P(1+\frac{0.05}{4})^{4t}$

$920 = P(1+\frac{0.05}{4})^{4 \times 1.5}$ $P = \frac{920}{(1.0125)^6} \approx 853.9208859$

$920 = P(1.0125)^6$

$\boxed{853.92}$ dollars

(c) (4 pts) A bacteria colony doubles its population every 20 minutes. If there are 3520 bacteria in 2 hours from now, how many will be in the colony 3 hours from now?

Let $B(k)$ = pop. size after k 20-minute periods.

$\boxed{\text{METHOD 1}}$ 2 hrs $\Leftrightarrow k=6$ 3 hrs $\Leftrightarrow k=9$ $B(6) \xrightarrow{\times 2} B(7) \xrightarrow{\times 2} B(8) \xrightarrow{\times 2} B(9)$, 2^3

$B(9) = B(6)\,2^3 = 3520 \times 2^3 = 28160$

$\boxed{\text{METHOD 2}}$ $B(k) = B(0)(2)^k$ $\rightarrow$ $B(9) = 55(2)^9 = 28160.$

$3520 = B(0)2^6$

$B(0) = \frac{3520}{2^6} = 55$

$\boxed{28160}$ bacteria

(d) (4 pts) How long does it take to double your principal in an account paying 7.3% per year, compounded monthly? (Round to two digits after the decimal).

$A(t) = P(1+\frac{0.073}{12})^{12t}$ $t = \frac{\ln(2)}{12\ln(1.006083)}$

$2\cancel{P} = \cancel{P}(1.00608\overline{3})^{12t}$ ≈ 9.524018796

$\ln(2) = \ln(1.00608\overline{3}^{12t})$

$\ln(2) = 12t\ln(1.00608\overline{3})$

$\boxed{9.52}$ years

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2. (16 points)

(a) (4 pts) Consider the sequence $S: 4, 10, 25, 62.5, \ldots$
Give the explicit formula for the sequence S and compute the 10th term.

Multiplicative with multiplier $m = 2.5$ $\left(\frac{10}{4} = \frac{25}{10} = \frac{62.5}{25} = 2.5\right)$

$$\boxed{S(k) = S(0)\,m^k = 4(2.5)^k}$$

$$S(10) = 4(2.5)^{10} \approx 38146.97266$$

EXPLICIT FORMULA: $\boxed{S(k) = 4(2.5)^k}$

$S(10) =$ $\boxed{38146.97266}$

(b) (4 pts) Consider the sequence $T: -10, -3.25, 3.5, 10.25, 17, \ldots$
Give the explicit formula for the sequence T and compute the 50th term.

Additive with increment $r = 6.75$ $\left(-3--10 = 3.5--3.25 = 10.25-3.5 \right.$ $\left. 17-10.25 = 6.75\right)$

$$\boxed{T(k) = T(0) + rk = -10 + 6.75k}$$

$$T(50) = -10 + 6.75(50) = 327.5$$

EXPLICIT FORMULA: $\boxed{T(k) = -10 + 6.75k}$

$T(50)$: $\boxed{327.5}$

(c) Fred invests \$30,000 in a bank account. The following sequence gives the amount in the bank account at the beginning of each year: $A: 30000, 33600, 37632, 45147.84, \ldots$

i. (5 pts) Give the explicit formula for the account value at t years and give the percent change over the first 10 years.

MULTIPLICATIVE! $\left(\frac{33600}{30000} = \frac{37632}{33600} = 1.12\right)$

$$\boxed{A(t) = A(0)\,m^t = 30000(1.12)^t}$$

PERCENT CHANGE OVER 10 YEARS $= \frac{\text{NEW} - \text{OLD}}{\text{OLD}} \times 100\% = \frac{30000(1.12)^{10} - 30000}{30000} \times 100\%$

$$= [(1.12)^{10} - 1] \times 100\% = 210.5848208\%$$

EXPLICIT FORMULA: $\boxed{A(t) = 30000(1.12)^t}$

percent change over first 10 years = $\boxed{210.58\%}$

ii. (3 pts) At what value of t does the account balance reach \$100,000?

$$100000 = 30000(1.12)^t$$
$$3.\overline{3} = (1.12)^t$$
$$\ln(3.\overline{3}) = \ln(1.12^t)$$
$$\ln(3.\overline{3}) = t \ln(1.12)$$

$$t = \frac{\ln(3.\overline{3})}{\ln(1.12)} \approx 10.62372515$$

t = $\boxed{10.62}$

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3. (16 points) Consider three different bank accounts.

Account A: annual interest rate of 6%, compounded continuously
Account B: annual interest rate of 6.2%, compounded quarterly
Account C: annual interest rate of $(r \times 100)\%$, compounded semi-annually

Round all final answers to two digits after the decimal point.

(a) (4 pts) What is the annual percentage yield (APY) for Accounts A and B?

$\boxed{A}$ $APY = [e^r - 1] \times 100\% = [e^{0.06} - 1] \times 100\% = 6.18365465$

$\boxed{B}$ $APY = [(1+\frac{r}{n})^n - 1] \times 100\% = [(1+\frac{0.062}{4})^4 - 1] \times 100\% = 6.34564532$

APY for Account A = $\boxed{6.18}$ %
APY for Account B = $\boxed{6.35}$ %

(b) (4 pts) What is the percentage change of Account A over any two year period (such as 0 to 2 years)? (~~Hint: You don't need to know the principal value to answer this~~.)

percent change from 0 to 2 $= \frac{\text{NEW} - \text{OLD}}{\text{OLD}} \times 100\% = \frac{Pe^{0.06 \times 2} - P}{P} \times 100\%$

$= [e^{0.06 \times 2} - 1] \times 100\% = 12.7496851\%$

percent change = $\boxed{12.75}$ %

(c) (4 pts) Harry invest P dollars into Account A and Ron invest \$1000 in Account B. At year 5, their account balances are the same. What is the value of P?

$H(t) = Pe^{0.06t}$ $R(t) = 1000\left(1+\frac{0.062}{4}\right)^{4t}$

$Pe^{0.06 \times 5} = 1000\left(1+\frac{0.062}{4}\right)^{4 \times 5}$ $P = \frac{1360.186797}{e^{0.3}}$

$Pe^{0.3} = 1360.186797$ ≈ 1007.651162

$P = \boxed{1007.65}$ dollars

(d) (4 pts) Frodo invests \$2000 into Account C. At $t = 3$ years, he gets a bank statement that tells him that his balance is exactly \$3500. What is the annual interest rate for Account C?

$A(t) = 2000\left(1+\frac{r}{2}\right)^{2t}$

$3500 = 2000\left(1+\frac{r}{2}\right)^{2 \times 3}$

$1.75 = \left(1+\frac{r}{2}\right)^6$

$(1.75)^{1/6} = 1 + \frac{r}{2}$

$1.097757319 = 1 + \frac{r}{2}$

$0.097757319 = \frac{r}{2}$

$r = 0.195514638$

$(r \times 100) = \boxed{19.55}$ %

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4. (19 points) You sell digit photo frames. Your total revenue and total cost (in dollars) for selling q photo frames are:

$$TR(q) = -0.4q^2 + 154q \qquad TC(q) = 53q + 525.$$

(a) (4 pts) Give the formulas for variable cost, average cost, average variable cost, and price.

$VC(q) = TC(q) - FC$ ← 525

$AC(q) = \frac{TC(q)}{q} = \frac{53q+525}{q}$

$AVC(q) = \frac{VC(q)}{q} = \frac{53q}{q}$

$TR(q) = pq \Rightarrow p = \frac{TR(q)}{q} = \frac{-0.4q^2+154q}{q}$

$VC(q) =$ $\boxed{53q}$

$AC(q) =$ $\boxed{53 + \frac{525}{q}}$

$AVC(q) =$ $\boxed{53}$

$p =$ $\boxed{-0.4q + 154}$

(b) (6 pts) Recall that $MR(q) = TR(q + 1) - TR(q)$ and $MC(q) = TC(q + 1) - TC(q)$.
Use these definitions to find the formulas for marginal revenue, $MR(q)$, and marginal cost, $MC(q)$. Simplify them as far as possible.

$MC(q) = TC(q+1) - TC(q) = [53(q+1) + 525] - [53q + 525]$
$= \cancel{53q} + 53 + \cancel{525} - \cancel{53q} - \cancel{525} = 53$

$MR(q) = TR(q+1) - TR(q) = [-0.4(q+1)^2 + 154(q+1)] - [-0.4q^2 + 154q]$
$= [-0.4(q^2+2q+1) + 154q + 154] + 0.4q^2 - 154q$
$= -\cancel{0.4q^2} - 0.8q - 0.4 + \cancel{154q} + 154 + \cancel{0.4q^2} - \cancel{154q}$
$= -0.8q + 153.6$

$MR(q) =$ $\boxed{-0.8q + 153.6}$

$MC(q) =$ $\boxed{53}$

(c) (5 pts) Find the quantity that maximizes profit and give the maximum profit value.

$\boxed{\text{METHOD 1}}$ $MR = MC \rightarrow -0.8q + 153.6 = 53 \Rightarrow -0.8q = -100.6$

$q = \frac{-101.6}{-0.8} = 125.75 \xrightarrow{\text{round up}} \boxed{126 \text{ frames}}$

$\boxed{\text{METHOD 2}}$ $\text{Profit} = P(q) = TR(q) - TC(q) = [-0.4q^2 + 154q] - [53q + 525]$
$= -0.4q^2 + 154q - 53q - 525 = -0.4q^2 + 101q - 525$

VERTEX: $q = -\frac{b}{2a} = -\frac{101}{2(-0.4)} = 126.25 \rightarrow \boxed{126 \text{ frames}}$ round to nearest

$P(126) = -0.4(126)^2 + 101(126) - 525$
$= 5850.60$

$q =$ $\boxed{126}$ frames

Max Profit = $\boxed{5850.60}$ dollars

(d) (4 pts) Between which two quantities is total revenue (TR) above \$10,000?

$10000 = -0.4q^2 + 154q$

$0 = -0.4q^2 + 154q - 10000$

$q = \frac{-154 \pm \sqrt{154^2 - 4(-0.4)(-10000)}}{2(-0.4)} =$ 82.69904372 and 302.3009563

from $q =$ $\boxed{82.699}$ to $q =$ $\boxed{302.301}$ frames

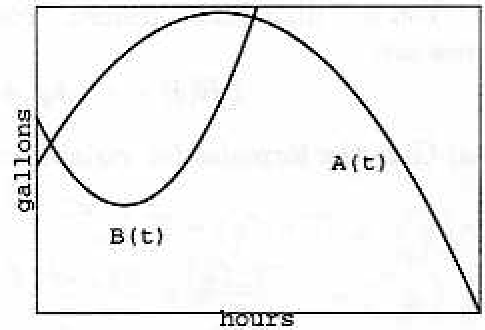
$\boxed{V1}$

5. (16 points)

The total amount (in gallons) of water in two vats at time, t, hours are given by

VAT A : $A(t) = -\frac{1}{2}t^2 + 5t + 12.$

VAT B : $B(t) = 1.25t^2 - 6t + 16.$



(a) (4 pts) Give the longest interval of time over which the amount in both vats is increasing.

Vertex of A : $t = -\frac{b}{2a} = -\frac{5}{2(-\frac{1}{2})} = 5$

A is increasing before $t = 5$

Vertex of B : $t = -\frac{b}{2a} = -\frac{-6}{2(1.25)} = 2.4$

B is increasing after $t = 2.4$

from $t =$ $\boxed{2.4}$ to $t =$ $\boxed{5}$ hours

(b) (4 pts) ~~How many gallons of water are in Vat B when it is at its lowest level~~?

Maximum difference when A has more?

$A(t) - B(t) = (-\frac{1}{2}t^2 + 5t + 12) - (1.25t^2 - 6t + 16)$

$= -1.75t^2 + 11t - 4$

VERTEX: $t = \frac{-b}{2a} = \frac{-11}{2(-1.75)} = 3.142857143$

NUMBER OF GALLONS $= A(3.142857143) - B(3.142857143)$

$= 13.28571429$

$\boxed{13.29}$ gallons

(c) (4 pts) Compute the average rate of change in Vat A from 1 hour to 4 hours. Include units for your answer.

$\frac{A(4) - A(1)}{4 - 1} = \frac{[-\frac{1}{2}(4)^2 + 5(4) + 12] - [-\frac{1}{2}(1)^2 + 5(1) + 12]}{3}$

$= 2.5$

average rate of change = $\boxed{2.5}$ units: $\boxed{\text{gallons/hour}}$

(d) (4 pts) Find all times when the amount of water in Vat A is equal to 5 gallons more than the amount of water in Vat B.

$-\frac{1}{2}t^2 + 5t + 12 = 1.25t^2 - 6t + 16 + 5$

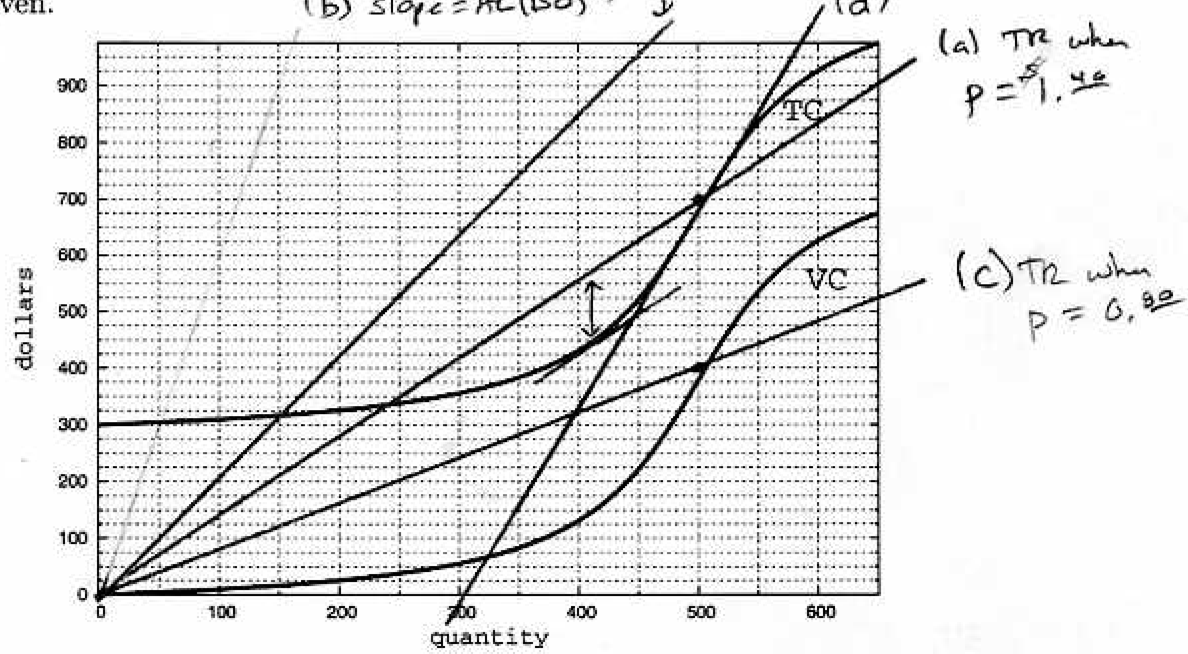
$-1.75t^2 + 11t - 9 = 0$

$t = \frac{-11 \pm \sqrt{11^2 - 4(-1.75)(-9)}}{2(-1.75)} =$ 0.9669219698, 5.318792316

$t =$ $\boxed{0.97 \text{ and } 5.32}$ hours

VI

6. (18 points) Harry sells mechanical pencils. The graphs of total cost and variable cost for producing pencils are given.



For each part, clearly explain your work in a sentence and label your work in the graph.

(a) Assume the market price is fixed at \$1.40 dollars per mechanical pencil.

i. (2 pts) Draw and label the total revenue (TR) graph that corresponds to this market price.

ii. (4 pts) Give the quantity at which profit is maximized and give the maximum profit value.

(0,0) (500, 700)

GAP = 575 − 450 = 125

$q =$ 405 mechanical pencils

Max Profit = 125 dollars

(b) (4 pts) Compute average cost (AC) at $q =$ 150

2 PTS (0,0) (250, 525)

$$\frac{525-0}{250-0} = 2.1$$

$AC(50) =$ 2.10 dollars per pencil

(c) (4 pts) If the market price is \$0.80 per pencil, find the largest interval over which some fixed costs are recovered? (Hint: First draw the total revenue graph that goes with this market price.)

(0,0) (500, 400) TR > VC ⇒ some FC recovered

from $q =$ 0 to $q =$ 505 mechanical pencils

(d) (4 pts) Find the quantity at which MC is largest and compute the marginal cost at this value.

POINTS: (500, 675)
(550, 850)

$$\frac{850-675}{550-500} = 3.5$$

$q =$ 500 mechanical pencils

$MC =$ 3.50 dollars per pencil