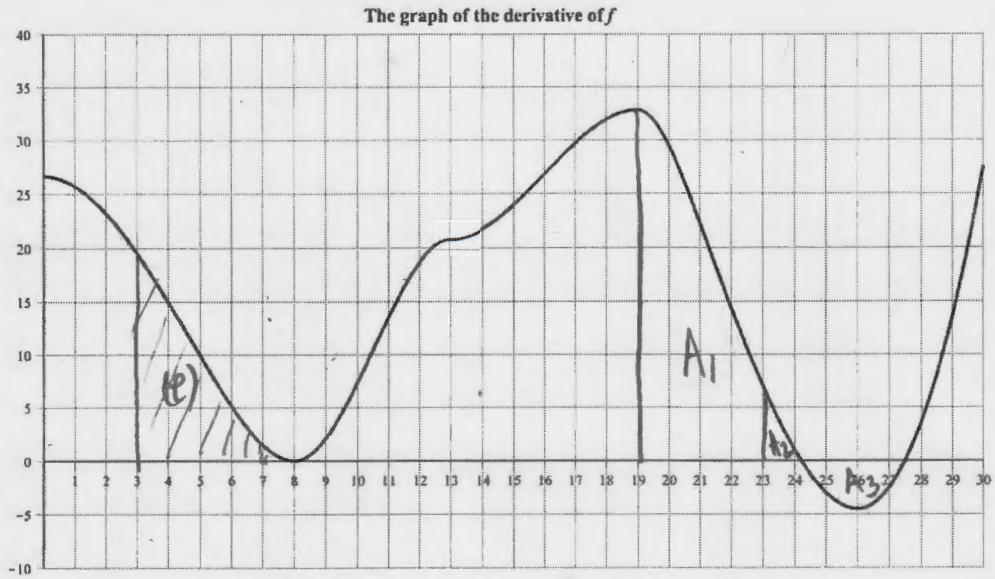
Solutions to the Final Exam

1. (12 points) Below is the graph of the **derivative** of $f(x)$. Answer the following questions about $f(x)$.



(a) In which interval(s) is the function f decreasing? Give your answer(s) in the form $a < x < b$.

f is decreasing when $f' < 0$: $24.2 < x < 27.5$

(b) In which interval(s) is the graph of the function f concave down? Give your answer(s) in the form $a < x < b$.

f is concave down when $f'' < 0$ or $f' \searrow$: $0 < x < 8$
$19 < x < 26$

(c) List all values of x for which $f(x)$ is a relative maximum.

when f' changes sign from + to − : $x \simeq 24.2$

(d) List all x-values where the graph of $f(x)$ has an inflection point.

when f'' changes sign or when f' changes direction : $x = 8, 19, 26$

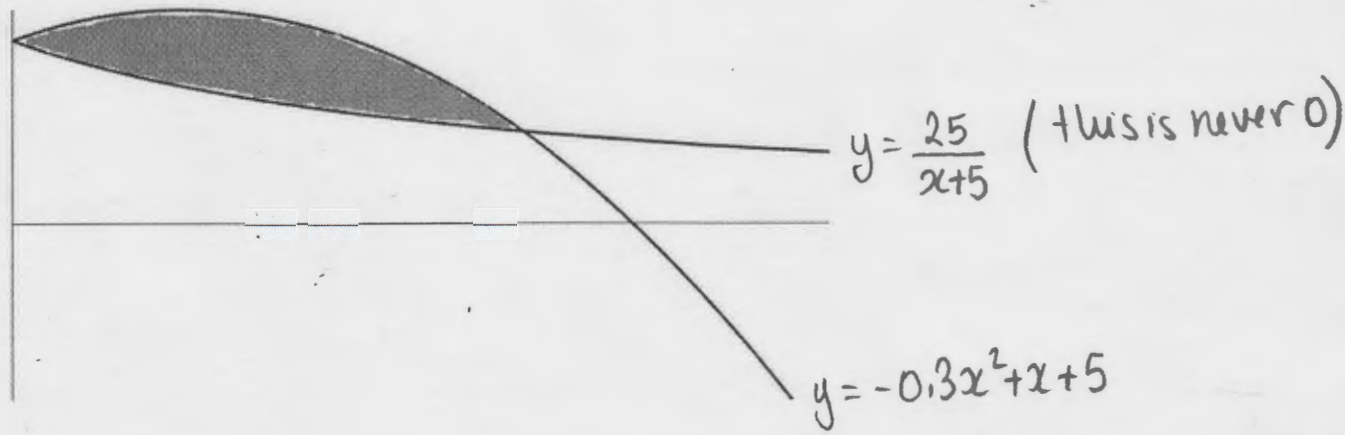
(e) Estimate the average rate of change of f from $x = 3$ to $x = 8$.

$$\frac{f(8)-f(3)}{8-3} = \frac{1}{5}\int_3^8 f'(t)\,dt \simeq \frac{1}{5}\cdot\frac{19\times 5}{2} = 9.5$$

(f) List the following three quantities from the smallest to the largest: $f(19)$, $f(23)$, $f(27.5)$. Give your answer in the form $f(a) < f(b) < f(c)$.

$f(19) < f(27.5) < f(23)$

2. (11 points) Below are the graphs of $y = \dfrac{25}{x+5}$ and $y = -0.3x^2 + x + 5$. Label the two graphs and find the shaded area. Simplify your answer and give it in exact form. Do not approximate using your calculator.



$$\frac{25}{x+5} = -0.3x^2 + x + 5$$

$$25 = (-0.3x^2 + x + 5)(x+5)$$

$$25 = -0.3x^3 - 1.5x^2 + x^2 + 5x + 5x + 25$$

$$0.3x^3 + 0.5x^2 - 10x = 0$$

$$(0.3x^2 + 0.5x - 10)x = 0$$

$$x = 0 \quad \text{OR} \quad x = \frac{-0.5 \pm \sqrt{0.25 - 4(0.3)(-10)}}{0.6}$$

$$= \frac{-0.5 \pm 3.5}{0.6} = 5$$

$$\text{Area} = \int_0^5 -0.3x^2 + x + 5 - \frac{25}{x+5}\,dx = -0.1x^3 + 0.5x^2 + 5x - 25\ln(x+5)\Big|_0^5$$

$$= -12.5 + 12.5 + 25 - 25(\ln 10 - \ln 5)$$

$$= 25 - 25\ln 2.$$

3. (14 points) Differentiate the following functions. You do not have to simplify your answers.

(a) $f(x) = e^{-0.2x}(4x-1)^5$

$$f'(x) = -0.2e^{-0.2x}(4x-1)^5 + e^{-0.2x} \cdot 5 \cdot (4x-1)^4 \cdot 4$$

(b) $g(t) = \ln\left(\sqrt{t+5}\right) + e^{\sqrt{t+4}}$ $= \frac{1}{2}\ln(t+5) + e^{(t+4)^{1/2}}$

$$g'(t) = \frac{1}{2(t+5)} + \frac{1}{2\sqrt{t+4}}e^{\sqrt{t+4}}$$

(c) Compute both f_x and f_y for $f(x,y) = \dfrac{2x-3y}{5x+6y} + \ln(xy)$.

$$f_x = \frac{2(5x+6y) - 5(2x-3y)}{(5x+6y)^2} + \frac{1}{x}$$

$$f_y = \frac{-3(5x+6y) - 6(2x-3y)}{(5x+6y)^2} + \frac{1}{y}$$

4. (11 points) A bakery makes Apple and Cherry pies. The profit from producing and selling x Apple pies and y Cherry pies is given by the joint profit function

$$P(x, y) = -2.5x^2 + 3xy - y^2 - 4x + 7y - 13,$$

where $P(x, y)$ is in dollars.

(a) This function is maximized at its critical point. What is the maximum profit

$3 \cdot (P_x = -5x + 3y - 4 = 0)$

$5 \cdot (P_y = 3x - 2y + 7 = 0)$

$+$

$-y - 12 + 35 = 0$

$23 = y$

$3x = 2y - 7 = 46 - 7 = 39$

$x = 13$

$P(13, 23) = -2.5(13)^2 + 3(13)(23) - 23^2 - 4 \cdot 13 + 7(23) - 13$

$= 41.5$ dollars

(b) The bakery has an order for 23 pies for Pi-Day. They have to make 10 Apple and 12 Cherry pies, but they are free to pick the type of the 23rd. Should they make an Apple pie or a Cherry pie? Use **partial derivatives** to explain your choice.

$P_x(10, 12) = -50 + 36 - 4 = -18$ Making one more Apple pie decreases profits.

$P_y(10, 12) = 30 - 24 + 7 = 13$

making one more Cherry pie increases profits.

They should make a Cherry pie.

5. (10 points) The demand function for a certain product is $p = 788 - 4x^2$ and the supply function is $p = 4x^2 + 80x + 20$.

(a) Find the equilibrium point.

$$788 - 4x^2 = 4x^2 + 80x + 20$$

$$0 = 8x^2 + 80x - 768$$

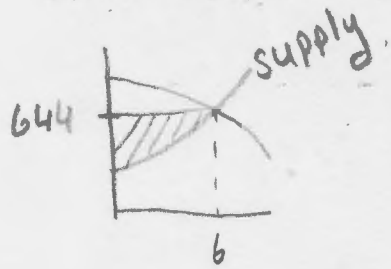
$$0 = x^2 + 10x - 96$$

$$x = \frac{-10 \pm \sqrt{100 - 4(-96)}}{2} = \frac{-10 \pm 22}{2} = 6$$

(the other value $-16 < 0$)

$$p(6) = 788 - 4 \cdot 36 = 644$$

(b) Find the producer's surplus.



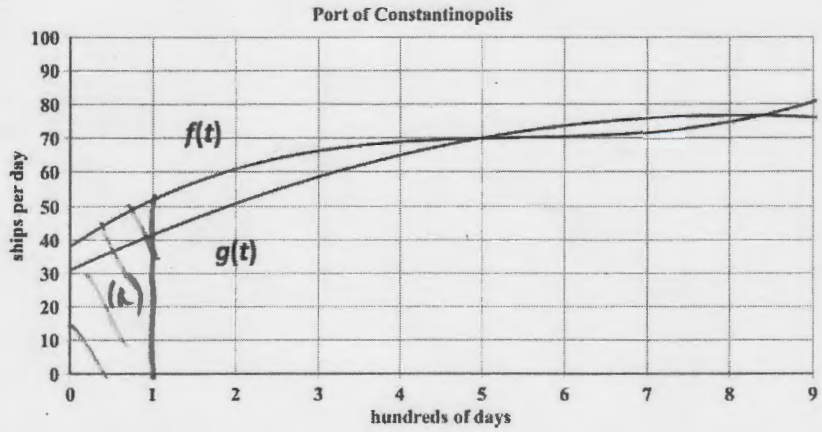
$$PS = 6 \cdot 644 - \int_0^6 4x^2 + 80x + 20 \, dx$$

$$= 3864 - \frac{4x^3}{3} + 40x^2 + 20x \Big|_0^6$$

$$= 3864 - 288 - 1440 - 120$$

$$= 2016$$

6. (14 points) The following are graphs showing the statistics for a 900 day interval for the port of Constantinapolis starting at the beginning of 1451. The number of ships coming in per day is given by $f(t) = 0.185t^3 - 3t^2 + 16.75t + 38$. The number of ships leaving the port is given by $g(t) = -0.7t^2 + 11.3t + 31$. Initially there were 35 ships at the port. **Be careful with units.**



(a) Use the graph to **estimate** the number of ships which have come into the port in the first 100 days. Show on the graph what you computed. Give units with your answer.

Area (a) $\simeq \frac{39+52}{2} \cdot 1 = 45.5$ hundred ships or 4550 ships

(b) Compute the average number of ships leaving the port per day during the first 200 days using the equations given. Round your answer to the nearest ship.

$$\frac{1}{2-0}\int_0^2 -0.7t^2+11.3t+31\,dt = \frac{1}{2}\left(-0.7\frac{t^3}{3} + 5.65t^2+31t\Big|_0^2\right)$$

$\simeq 41$ ships/day

(c) Say in words what the quantity $\int_1^2 (f(t) - g(t))\,dt$ represents. Do not compute it.

The increase in the number of ships at port from day 100 to day 200.

(d) Use the graph to estimate when the number of ships at port reaches its maximum value. Use integration to find the total number of ships at port that day. Round your answer to the nearest ship.

$t=5$

$$0.35 + \int_0^5 0.185t^3 - 3t^2 + 16.75t + 38 + 0.7t^2 - 11.3t - 31\,dt$$

$$= 0.35 + 0.185\frac{t^4}{4} - 2.3\frac{t^3}{3} + 5.45\frac{t^2}{2} + 7t\Big|_0^5 \simeq 36.55 \text{ hundred ships}$$

$= 3655$ ships

7. (17 points) You produce and sell Chorks. The total revenue, in **hundreds** of dollars, for selling x-**hundred** Chorks is given by the function $TR(x)$. You are not given the formula for $TR(x)$, but you know that, for any value of h,

$$TR(x+h) - TR(x) = 4.95h - 0.1xh - 0.05h^2.$$

(a) Compute the Marginal Revenue $MR(x)$ and the Total Revenue $TR(x)$.

$$MR(x) = \lim_{h\to 0} \frac{4.95h - 0.1xh - 0.05h^2}{h} = \lim_{h\to 0} 4.95 - 0.1x - 0.05h$$
$$= 4.95 - 0.1x$$

$$TR(x) = 4.95x - 0.05x^2$$

(b) Your Marginal Cost is given by $MC(x) = 3 + \dfrac{1}{x+1}$ and your fixed costs are \$1000. What is your Total Cost function $TC(x)$ in **hundreds** of dollars where x is in **hundreds** of Chorks? Be careful with the units.

$$TC(x) = 3x + \ln(x+1) + C$$
$$10 = TC(0) = 0 + \ln 1 + C = C$$
$$TC(x) = 3x + \ln(x+1) + 10$$

(c) Find all positive values of x at which the graphs of TR and TC have parallel tangent lines. Do these value(s) of x give a local minimum or local maximum for the Profit function? Explain.

when $MR = MC$

$$4.95 - 0.1x = 3 + \frac{1}{x+1} \longrightarrow 1.95 - 0.1x = \frac{1}{x+1}$$

$$1 = (x+1)(1.95 - 0.1x) = 1.95x - 0.1x^2 + 1.95 - 0.1x$$

$$0.1x^2 - 1.85x - 0.95 = 0 \longrightarrow x = \frac{1.85 \pm \sqrt{1.85^2 - 4(0.1)(-0.95)}}{0.2}$$

$$= \frac{1.85 \pm 1.95}{0.2} = 19$$

when $x < 19$, say $x = 9$, $MC(9) = 3.1$, $MR(9) = 4.95 - 0.9 = 4.05 < MC$ so profits ↗

when $x > 19$, say $x = 49$, $MC(49) = 3.02$, $MR(49) = 4.95 - 4.9 = 0.05 < MC$ so profits ↘

$x = 19$ is a local max for profit

(d) Find the maximum profit. Round your answer to the nearest dollar.

$$P(19) = TR(19) - TC(19) = [4.95(19) - 0.05(19)^2] - [3 \cdot 19 + \ln(19+1) + 10]$$
$$= 1.95(19) - 0.05(19)^2 - \ln 20 - 10$$
$$= 9 - \ln 20$$
$$\simeq 6.00 \text{ hundred dollars} = \$600$$

8. (11 points)

(a) Evaluate $\int \frac{\sqrt{t}+5t^3}{t} - \ln\left(e^{4t}\right)\, dt$.

$$= \int t^{-1/2} + 5t^2 - 4t \, dt = 2t^{1/2} + \frac{5}{3}t^3 - 2t^2 + C$$

(b) A new financial company opens up and expects its monthly rate of income flow to be given by $r(t) = 12000e^{0.02t}$ dollars per month, where t is months after it opened. Find the total income in the first 3 years of operation. Round your answer to the nearest dollar.

$$\text{Total Income} = \int_0^{36} 12000e^{0.02t}\, dt = \frac{12000}{0.02} e^{0.02t}\Big|_0^{36}$$

$$= 600{,}000\left(e^{0.72} - 1\right)$$

$$\approx \$632{,}660$$