MATH 112 FINAL EXAM WINTER 2016 SOLUTIONS

1 (16 pts) Compute the indicated derivatives. DO NOT SIMPLIFY. Box your final answer.

a) $f(t) = \sqrt{\ln(t^2 - 3t) + 7}$

$$f'(t) = \frac{1}{2}\left(\ln(t^2-3t)+7\right)^{-1/2}\left(\frac{1}{t^2-3t}\right)(2t-3)$$

b) $u = \dfrac{e^x \ln x}{x^2+\frac{1}{x}-7}$

$$\frac{du}{dx} = \frac{\left(e^x \ln x + e^x \frac{1}{x}\right)\left(x^2+\frac{1}{x}-7\right) - \left(e^x \ln x\right)\left(2x - \frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x}-7\right)^2}$$

c) $z = 2e^y x + \frac{y}{x} + \ln(xy^2) + x$

$$\frac{\partial z}{\partial x} = 2e^y - \frac{y}{x^2} + \frac{1}{xy^2}\cdot y^2 + 1$$

$$\frac{\partial z}{\partial y} = 2e^y x + \frac{1}{x} + \frac{1}{xy^2}\left(x(2y)\right)$$

$\boxed{2}$ (6 pts) Suppose we do not have a formula for a certain function $f(x)$, but we know that:

$$f(m+h) - f(m) = \frac{12h}{(2+m+h)(5+m)}$$

Compute $f'(3)$. Show all steps clearly.

$$\frac{f(3+h) - f(3)}{h} = \frac{\left(\frac{12h}{(2+3+h)(5+3)}\right)}{h} = \left(\frac{12\cancel{h}}{(5+h)(8)}\right) \cdot \frac{1}{\cancel{h}}$$

$$f'(3) = \lim_{h \to 0} \left(\frac{12}{(5+h)8}\right) = \frac{12}{(5)(8)} = \frac{3}{10} = 0.3$$

ANSWER: $f'(3) =$ 0.3

$\boxed{3}$ (10 pts) Compute each of the following **integrals**. SIMPLIFY and $\boxed{\text{box}}$ your final answers.

a) $\int \frac{3}{x^2} - 2e^{2x} + \frac{7x^2+3}{x}\, dx$

$$= \int 3x^{-2} - 2e^{2x} + 7x + \frac{3}{x}\, dx$$

$$= 3\frac{x^{-1}}{-1} - 2\frac{e^{2x}}{2} + 7\frac{x^2}{2} + 3\ln(x) + C$$

$$= \boxed{-\frac{3}{x} - e^{2x} + \frac{7}{2}x^2 + 3\ln(x) + C}$$

b) $\int_9^{25} \frac{3}{\sqrt{t}} + 2\, dt$

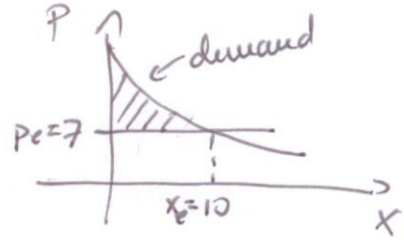
$$= \int_9^{25} 3t^{-1/2} + 2\, dt$$

$$= \left(3\frac{t^{1/2}}{(1/2)} + 2t\right)\Big|_9^{25} = \left(6\sqrt{t} + 2t\right)\Big|_9^{25}$$

$$= \left[6\sqrt{25} + 2(25)\right] - \left[6\sqrt{9} + 2(9)\right]$$

$$= 80 - 36 = \boxed{44}$$

4 (8 pts) The demand and supply functions for a product are:



$$\text{demand:} \quad p = \frac{77}{x+1}$$

$$\text{supply:} \quad p = 2 + 0.5x$$

where p is the price per unit, in dollars, and x is the number of units.

Compute the consumers surplus under pure competition.

1. Find market equilibrium: $\frac{77}{x+1} = 2+0.5x$

$77 = (2+0.5x)(x+1)$

$77 = 2x + 2 + 0.5x^2 + 0.5x$

$0.5x^2 + 2.5x - 75 = 0.$

$x^2 + 5x - 150 = 0$

QF. or factor $(x+15)(x-10) = 0.$

$\boxed{x_e = 10}$, $\cancel{x = -15}$

$p = 2 + 0.5(10) \Rightarrow \boxed{p_e = 7}$.

2. $CS = \int_0^{10} \left(\frac{77}{x+1}\right) - (7)\, dx$

$= \left(77 \ln(x+1) - 7x\right)\Big|_0^{10}$

$= \left(77 \ln(11) - 70\right) - \left(77 \underbrace{\ln(1)}_{0} - 7(0)\right)$

$= 77 \ln(11) - 70$

$= 114.6379\ldots$

ANSWER: Consumers Surplus = \$ 114.64

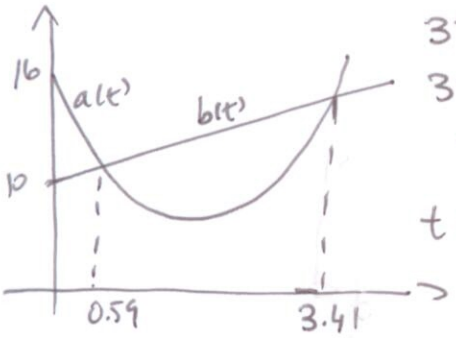
(You may round your final answer to the nearest two digits)

5 (12 pts) Two bicyclists, Anne and Bob, are next to each other at time $t = 0$, and travel along the same straight road. Their respective speeds at t hours are given by the functions:

Biker Anne's speed: $\boldsymbol{a(t) = 3t^2 - 10t + 16}$ miles/hour

Biker Bob's speed: $\boldsymbol{b(t) = 2t + 10}$ miles/hour

a) At what time during the first 1.5 hours are the two bikers farthest apart?



$$3t^2 - 10t + 16 = 2t + 10$$
$$3t^2 - 12t + 6 = 0$$
$$t^2 - 4t + 2 = 0$$
$$t = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2} \begin{cases} 3.414\ldots \\ 0.5857\ldots \end{cases}$$

Answer: at $t =$ 0.59 hours.

b) Which biker is ahead after 1 hour, and by how much? Show work.

Method 1: dist. between (with A ahead) if positive $= \int_0^1 a(t) - b(t)\,dt = \int_0^1 (3t^2 - 10t + 16) - (2t + 10)\,dt$

$= \int_0^1 3t^2 - 12t + 6\,dt = (t^3 - 6t^2 + 6t)\Big|_0^1 = 1$ mile

Method 2:

position of A: $A(t) = t^3 - 5t^2 + 16t$ (relative to starting place)

position of B: $B(t) = t^2 + 10t$ (— " —)

Distance between after 1 hr: $A(1) - B(1) = (1 - 5 + 16) - (1 + 10) = 12 - 11$

Answer: Biker A is ahead by 1 miles.

c) Recall that the instantaneous speed for Biker Bob is given by the linear function: $b(t) = 2t + 10$. Compute the **average speed** of Biker Bob over the time interval from $t = 1$ to $t = 2.5$ hours.

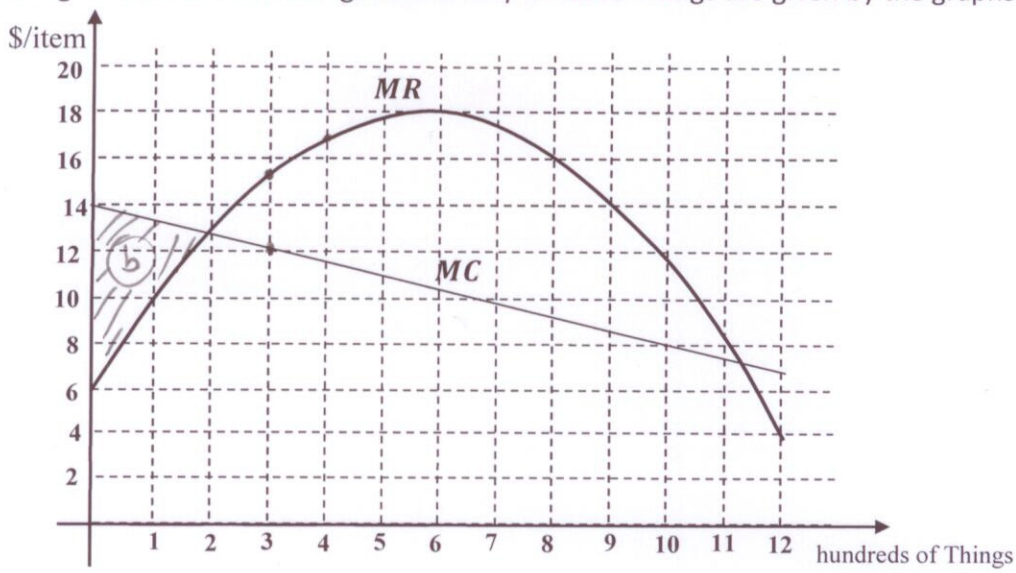
Change in distance for Bob from $t = 1$ to $t = 2.5$ hrs is

$$\Delta B = \int_1^{2.5} b(t)\,dt = \int_1^{2.5} (2t + 10)\,dt = (t^2 + 10t)\Big|_1^{2.5} = 31.25 - 11$$
$$= 20.25 \text{ miles}$$

$$\text{Average speed} = \frac{\Delta B}{\Delta t} = \frac{20.25 \text{ miles}}{1.5 \text{ hrs.}} = 13.5 \text{ mph}$$

Answer: Bob's average speed was 13.5 miles per hour.

6 (12 pts) The marginal revenue and marginal cost at q hundred Things are given by the graphs below.



You also know that your fixed costs are 2 hundred dollars.

a) Estimate your Total Cost for producing 300 Things. Show your work.

$TC(3) = [\text{area under MC from 0 to 3}] + FC \cong 14 \square 12 + 2 = \frac{1}{2}(14+12)3 + 2 = 39 + 2$

Answer: $TC(3) \approx$ 41 hundred dollars

b) Estimate the minimal profit (maximal loss), and the quantity at which it occurs. Show work.

$MC > MR$ from $q=0$ to $q \cong 1.9$ so profit ↘ until $q \cong 1.9$, then ↗ until $q \cong 11.3$

Min Profit is at $q \cong 1.9$

Value $= -[\text{area between MC and MR from } q=0 \text{ to } q=1.9] - FC = -\triangle - 2 \cong -7.6 - 2$

Answer: Min Profit $\approx$ -9.6 hundred dollars, at $q \approx$ 1.9 hundred Things

ok: -10 $\quad$ 2

c) Estimate the change in revenue from $q = 3$ to $q = 4$ hundred Things. Show work.

$TR(4) - TR(3) = \int_3^4 MR(q)\,dq = \text{area under MR} \cong \frac{1}{2}(15+17)(1) = 16$

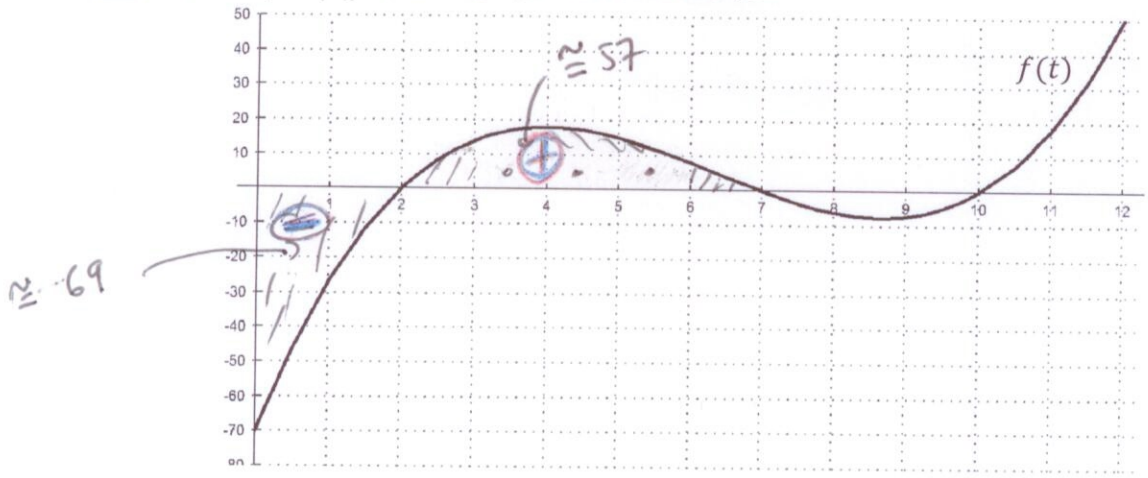
Answer: 16 hundred dollars

d) Does your profit increase or decrease if you produce and sell the 301st Thing? By approximately how much?

$MP(3) = MR(3) - MC(3) \cong 15.3 - 12.1 = 3.2$ \$

Answer: The profit increases/decreases (circle one) by about 3.2 dollars

7 (10 pts) The following is the **graph of a function** $f(t)$.



Let $A(m) = \int_0^m f(t)\,dt$ be the accumulated graph of $f(t)$. Answer the following questions. Read each question carefully!

a) For each part below, circle the correct answer. No need to justify.

i.	The value of $f(5)$ is	POSITIVE,	NEGATIVE,	or	ZERO.
ii.	The value of $f'(5)$ is	POSITIVE,	NEGATIVE,	or	ZERO
iii.	The value of $f''(5)$ is	POSITIVE,	NEGATIVE,	or	ZERO
iv.	The value of $A(7)$ is	POSITIVE,	NEGATIVE,	or	ZERO
v.	The value of $A'(7)$ is	POSITIVE,	NEGATIVE,	or	ZERO

b) Find the longest interval during which the derivative $f'(t)$ is **decreasing**.

when $f'' < 0$ so when f is concave down

Answer: from $t =$ 0 to $t =$ 7

c) Estimate $\boldsymbol{A'(9)}$.

$A'(9) = f(9) \cong -8$

Answer: $A'(9) \approx$ -8

d) $f(t)$ has inflection points at $x =$ 7 *(list all, no need to justify)*

e) The local minima of $\boldsymbol{A(m)}$ are at $m =$ 2, 10 *(list all, no need to justify)*

when $f(t) = A'(t)$ is zero changing from $-$ to $+$.

8 (14 Points) You produce and sell flat-screen TV's and Blu-ray Players.

(a) (2 pts) Suppose you sell each TV for \$2000 and each Player for \$500. Give a formula for the total revenue $R(x, y)$, in dollars, which results from selling x TV's and y Players.

ANSWER: $R(x, y) =$ $2000x + 500y$

(b) Suppose your profit from selling x TV's and y Players is given by the function:

$$\boldsymbol{P(x, y) = 0.1x^2 + 0.1\,y^2 - 0.6xy + 300x + 100y - 1000}$$

i. (2 pts) Compute the two partial derivatives of your profit function.

$P_x(x, y) =$ $0.2x - 0.6y + 300$

$P_y(x, y) =$ $0.2y - 0.6x + 100$

ii. (6 pts) Find all candidates (x, y) for local minima or maxima of the profit $P(x, y)$.

$$\begin{cases} 0.2x - 0.6y + 300 = 0 \\ 0.2y - 0.6x + 100 = 0 \end{cases}$$

$$\begin{cases} x - 3y + 1500 = 0 \\ y - 3x + 500 = 0 \end{cases} \Rightarrow \boxed{y = 3x - 500}$$

$x - 3(3x - 500) + 1500 = 0$

$x - 9x + 1500 + 1500 = 0$

$-8x + 3000 = 0$

$8x = 3000$

$x = \frac{3000}{8}$

$x = 375$

$y = 3(375) - 500 = 625$

Answer: $(x, y) =$ $(375, 625)$

iii. (4 pts) Suppose you've produced and sold 300 TV's and 250 Players. Use a partial derivative to estimate the increase in your profit if you sell one more TV. Show your work, clearly.

$\Delta P \cong P_x(300, 250) = 0.2(300) - 0.6(250) + 300 = 210$

Answer: Profit will change by about \$ 210

9 (12 pts) The Demand Curve for selling Items has the formula:

$$p = 1 - 0.2\sqrt{q},$$

where the quantity q is in hundreds of Items and the price p is in dollars per Item.

The total cost (in hundreds of dollars) to produce q hundred Items is given by the formula:

$$TC(q) = 0.01q + 0.5.$$

Let $\boldsymbol{P(q)}$ denote the **profit** (in hundreds of dollars) you earn by producing and selling q hundred Items.

a) Determine the formula for the **profit** $P(q)$, as an expression in q. Simplify your answer.

$TR(q) = pq = (1-0.2\sqrt{q})q = q - 0.2\sqrt{q}\cdot q = q - 0.2q^{3/2}$

$P(q) = TR(q) - TC(q) = (q - 0.2q^{3/2}) - (0.01q + 0.5)$

ANSWER: $P(q) =$ $-0.2q^{3/2} + 0.99q - 0.5$

b) Compute the critical number(s) of the profit.

$P'(q) = -0.2\,\frac{3}{2}\,q^{1/2} + 0.99 = 0.$

$-0.3\sqrt{q} + 0.99 = 0$

$0.3\sqrt{q} = 0.99$

$\sqrt{q} = 3.3$

$q = (3.3)^2$

ANSWER: $q =$ 10.89 hundred Items

c) Use the Second Derivative Test to determine whether each critical number you found above gives a local maximum or a local minimum for the profit function, $P(q)$. Show work clearly, and box your answer(s).

$P''(q) = (-0.3q^{1/2} + 0.99)' = -0.3\,\frac{1}{2}\,q^{-1/2} = \frac{-0.3}{2\sqrt{q}}$

$P''(10.89) = \frac{-0.3}{2(3.3)} < 0$

By the 2nd derivative test, $\boxed{q = 10.89 \text{ is a local } \underline{\text{max}}. \text{ for the Profit}}$