1. (10 points) Compute the indicated derivatives. Do NOT simplify your answers, but BOX your final answers.

(a) If $f(x) = [x^3(x^2+3x)]^4$, compute $f'(x)$.

Method I: Generalized Power Rule first:

$$f'(x) = 4\left[x^3(x^2+3x)\right]^3 \left[x^3(x^2+3x)\right]' \quad \text{(Product Rule here)}$$

$$= \boxed{4\left[x^3(x^2+3x)\right]^3 \left[(3x^2)(x^2+3x) + x^3(2x+3)\right]}$$

(OR) Method II: Distribute the power first, then use Product Rule

$$f(x) = x^{12}(x^2+3x)^4$$

$$f'(x) = (x^{12})'(x^2+3x)^4 + x^{12}\left((x^2+3x)^4\right)' \quad \text{Gen. Power Rule here}$$

$$= \boxed{(12x^{11})(x^2+3x)^4 + x^{12}\left(4(x^2+3x)^3(2x+3)\right)}$$

(b) If $y = \dfrac{5}{2x^3+7}$, compute $\dfrac{d^2y}{dx^2}$. ← $\underline{\underline{2^{nd}}}$ derivative!

Method I: Quotient Rule, twice

$$\frac{dy}{dx} = \frac{(0)(2x^3+7) - 5(6x^2)}{(2x^3+7)^2} = \frac{-30x^2}{(2x^3+7)^2}$$

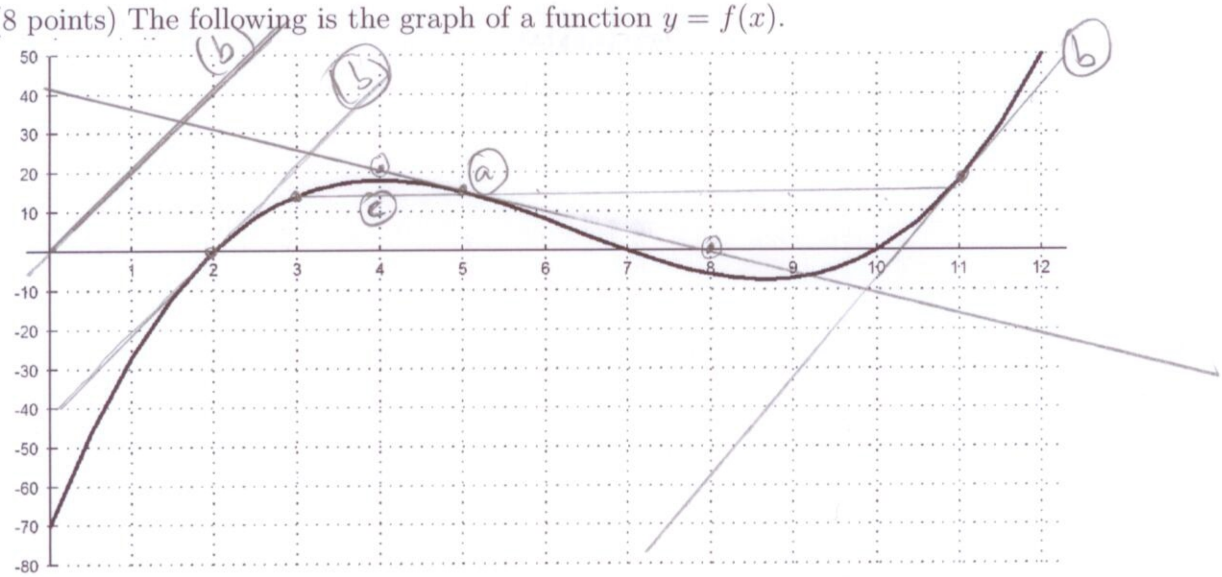
$$\frac{d^2y}{dx^2} = \boxed{\frac{-30(2x)(2x^3+7)^2 - (-30x^2)\left(2(2x^3+7)(6x^2)\right)}{\left((2x^3+7)^2\right)^2}}$$

Method II: $y = 5(2x^3+7)^{-1}$

$$\frac{dy}{dx} = 5(-1)(2x^3+7)^{-2}(6x^2) = (-30x^2)(2x^3+7)^{-2}$$

$$\frac{d^2y}{dx^2} = \boxed{(-60x)(2x^3+7)^{-2} + (-30x^2)(-2)(2x^3+7)^{-3}(6x^2)}$$

2. (8 points) The following is the graph of a function $y = f(x)$.



Use this graph to answer the following questions.
DRAW and LABEL on the graph any lines you use, and be as precise as possible.

(a) Estimate the value of $\frac{f(5.001) - f(5)}{0.001}$. ← $\cong$ slope of tangent line at $x=5$

$\cong \frac{0-20}{8-4} = \frac{-20}{4}$

ANSWER: $\frac{f(5.001) - f(5)}{0.001} \approx \boxed{-5}$ (or near values)

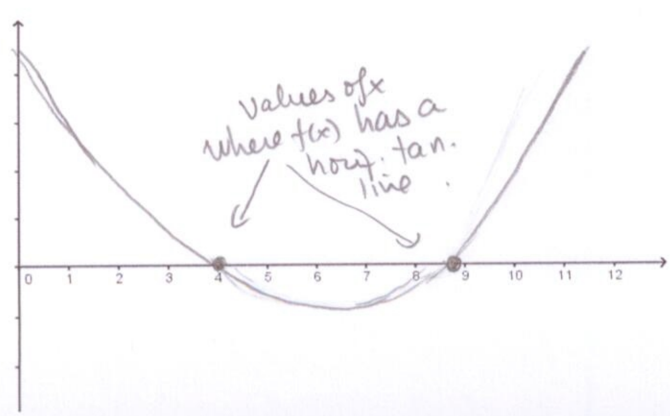
(b) Find **all** values of x where $\underbrace{f'(x) = 20}_{\text{slope of tangent line is} = 20.}$

ANSWER: At $x = \boxed{2, 11}$ (or near values)

(c) Find a positive value h such that $\underbrace{\frac{f(3+h) - f(3)}{h}}_{\text{slope of secant line}} = 0$.

from $x=3$ to $x=3+h=5$ ANSWER: $h =$ 2 (or $\cong 7.8$ if 2nd pt is $x=3+h=10.8$)

(d) Sketch the graph of $f'(x)$. I'm only looking for rough shape and correct x-intercepts.



3. (13 points)

(a) (8 pts) A car drives on a straight road. Its distance from a certain point is given by a function $D(t)$, where the time t is in seconds and the distance $D(t)$ is in feet.
We don't have the formula for $D(t)$, but we know that from time $t = 1$ to time $t = 1+h$ seconds, the change in distance for this car is given by the formula:

$$D(1+h) - D(1) = \left(h^2 + 3h\right)\sqrt{1+h}$$

i. Compute the car's **average speed** over the interval from $t = 1$ to $t = 4$ seconds.

$D(4) - D(1) = (3^2 + 3\cdot 3)\sqrt{1+3} = 18\sqrt{4} = 36$

$1+h=4$

$h=3$

avg speed $= \frac{D(4)-D(1)}{4-1} = \frac{36}{3} = 12$

ANSWER: $\boxed{12}$ feet per second

ii. Compute the car's **instantaneous speed** at $t = 1$ seconds. Show all steps.

$\frac{D(1+h)-D(1)}{h} = \frac{(h^2+3h)\sqrt{1+h}}{h} = (h+3)\sqrt{1+h}$

$D'(1) = \lim_{h\to 0}\left((h+3)\sqrt{1+h}\right) = 3\sqrt{1} = 3$

ANSWER: $\boxed{3}$ feet per second

(b) (5 pts) Suppose $d(t) = t\sqrt{t^2+7}$ is the distance an object traveled, in meters, after t minutes. Compute this object's $\underbrace{\text{instantaneous speed at } t = 3 \text{ minutes}}_{\text{i.e. } d'(3)}$.

$d'(t) = 1\sqrt{t^2+7} + t\left(\sqrt{t^2+7}\right)'$

$= \sqrt{t^2+7} + t\,\frac{1}{\cancel{2}}(t^2+7)^{-1/2}(\cancel{2}t)$

$= \sqrt{t^2+7} + \frac{t^2}{\sqrt{t^2+7}}$

$d'(3) = \sqrt{9+7} + \frac{9}{\sqrt{9+7}}$

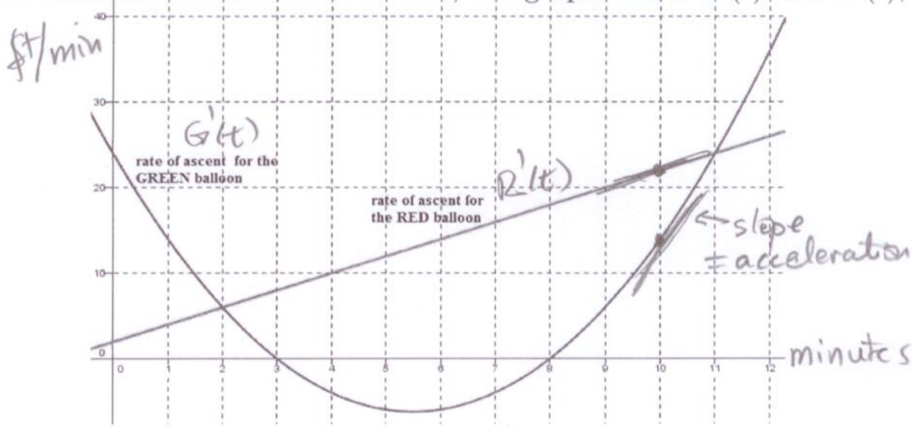
$= 4 + \frac{9}{4}$

$= 4 + 2.25$

ANSWER: 6.25 meters per minute

(or $\frac{25}{4}$)

4. (6 points) A Red balloon and a Green balloon rise and fall. When we start watching, at $t = 0$, both balloons are 50 feet above the ground. Their altitudes at time t minutes are given by functions $R(t)$ and $G(t)$, both measured in feet. The graphs below show the instantaneous rates of ascent of the balloons. That is, the graphs show $R'(t)$ and $G'(t)$, in feet per minute.



In this problem you do not have to show work or explain your answers.

(a) For each of the following statements circle the correct answer: True (T), False (F), or cannot tell based on the given information (CT).

At $t = 2$ min, Green balloon's altitude is higher than 50 feet. **T** F CT

(because $G' > 0$ from $t=0$ to $t=2$, so $G(t) \uparrow$)

At $t = 2$ min, the Green balloon is higher than the Red one. **T** F CT

($G'(t) > R'(t)$ the whole time, so Green is higher.)

The distance between the balloons is greater at $t = 3$ than at $t = 2$ T **F** CT

(Past $t=2$, $R' > G'$ so Red is catching up)

At $t = 10$ min, the Green balloon is rising faster than the Red one. T **F** CT

$G'(10) > R'(10)$ ← not true.

At $t = 10$ min, Green balloon's acceleration is greater than the Red's. **T** F CT

slope of tan. line to graph of Green's rate at $t=10$.

slope of tan to $R'(t)$ at $t=10$.

(b) Find the longest time interval during which the Green balloon is descending.

$G(t) \downarrow$ when $G'(t) < 0$ (below x-axis)

ANSWER: From $t =$ 3 to $t =$ 8 minutes

5. (13 points) You sell Items. The formulas for the total revenue and total cost, in **hundreds of dollars**, for selling and producing q **hundred Items** are:

total revenue: $TR(q) = 30q$ total cost: $TC(q) = q^3 - 15q^2 + 78q + 10$

(a) What is the marginal cost at 2 hundred items? Include correct units, and interpretation.

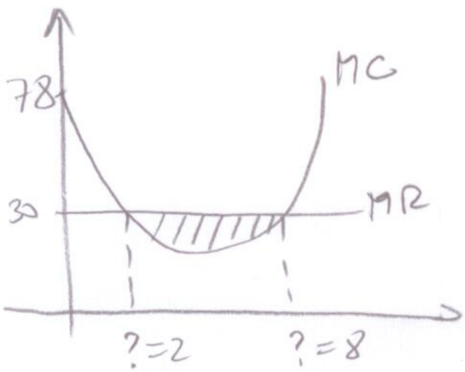
$MC(q) = TC'(q) = 3q^2 - 30q + 78$

$MC(2) = 3(2)^2 - 30(2) + 78 = 30$

ANSWER: The marginal cost at 200 items is: 30 Units: \$ (or \$/item)

Interpretation/meaning: the 201^{st} item costs an extra \$30 to produce

(b) Find the longest interval on which marginal revenue exceeds marginal cost.
(HINT: Sketch the graphs of MR and MC on the same set of axes.)



$MR = TR' = 30$

$MC = TC' = 3q^2 - 30q + 78$

$MR = MC: \quad 30 = 3q^2 - 30q + 78$

$3q^2 - 30q + 48 = 0$

$q^2 - 10q + 16 = 0$

Q.F. (or factor) $q = 2$ or $q = 8$.

ANSWER: From $q =$ 2 to $q =$ 8 hundred Items

(c) What is the maximum value of profit? (Include units)

Profit increases as long as $MR > MC$, so from $q=2$ to $q=8$.

So, it will be <u>max</u> at $q = 8$.

$TR(8) = 30(8) = 240$ hundred \$

$TC(8) = 8^3 - 15(8)^2 + 78(8) + 10 = 186$ hundred \$

$P(8) = TR(8) - TC(8)$

$= 240 - 186$

$= 54$

ANSWER: Maximum profit is: 54 Units: hundred \$