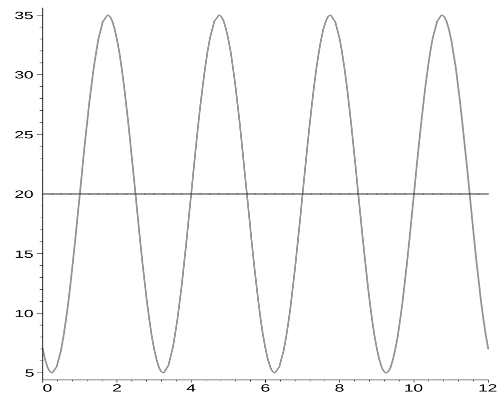
Exam Two Solutions

1 (a) The graph is given below. To find the amplitude, period, phase shift, and the mean (vertical shift), see part (b), below.



(The horizontal line shows the mean at $y = 20$.)

(b) We're considering the depth of the squid. The largest (maximum) value is 35 feet, the smallest (minimum) value is 5 feet, so the mean is $D = (35+5)/2 = 20$ feet. The amplitude is therefore $A = \max - D = 35 - 20 = 15$ (or $A = D - \min = 20 - 5 = 15$) feet. The squid goes through two cycles (periods) in 6 minutes, so the period is $B = 3$ minutes. Finally, we're told that the time at the maximum is $t_{\max} = 1.75$ minutes (that's 1 minute, 45 seconds), so one possible phase shift is $C = t_{\max} - B/4 = 1$ minute. Thus our formula is

$$d(t) = 15 \sin\left(\frac{2\pi}{3}(t-1)\right) + 20.$$

(c) This question simply asks for $d(5)$. Computation shows that this is $d(5) = 15\sin(8\pi/3) + 20$ feet, or $20 + 15\sqrt{3}/2$ feet, or roughly 32.990 feet.

(d) Now we are asked to find the first time after $t = 1$ with $d(t) = 8$ feet. Thus,

$$15 \sin\left(\frac{2\pi}{3}(t-1)\right) + 20 = 8,$$

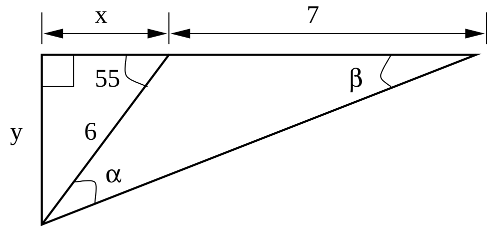
or

$$\sin\left(\frac{2\pi}{3}(t-1)\right) = -0.8.$$

We've reduced the problem to solving $\sin(\theta) = -0.8$ for θ, then setting $\theta = \frac{2\pi}{3}(t-1)$ and solving for t.

First, let's find what solutions there are for $\sin(\theta) = -0.8$. The first is the principal solution, $\theta_1 = \sin^{-1}(-0.8) \approx -0.9272952$ radians. The second, the symmetry solution, is $\theta_2 = \pi - \sin^{-1}(-0.8) \approx 4.0688879$ radians. These correspond to $t_1 = 1 + \frac{3}{2\pi}\sin^{-1}(-0.8) \approx 0.557$ minutes and $t_2 = 1 + \frac{3}{2\pi}(\pi - \sin^{-1}(-0.8)) \approx 2.943$ minutes. All other solutions follow from these two by adding multiples of the period (which is $B = 3$ here) to t_1 or t_2. Thus the first solution after $t = 1$ is $t \approx 2.943$ minutes.

2 (a) To find α and β (in degrees) in the triangle below, we begin by finding the sides x and y that I've labeled in the drawing.



By looking at the small triangle, we can see that $\cos(55°) = x/6$ and $\sin(55°) = y/6$, so

$$x = 6\cos(55°) \approx 3.4414586 \qquad \text{and} \qquad y = 6\sin(55°) \approx 4.9149123.$$

From the big triangle, we can see that $\tan(\beta) = y/(7+x)$, so

$$\beta = \tan^{-1}\left(\frac{6\sin(55°)}{7+6\cos(55°)}\right) \approx 25.207°.$$

There are then several ways to get α. The easiest is to notice that $\alpha + \beta = 55°$, so that $\alpha = 55° - \beta \approx 29.793°$.

(b) From the picture, we know that $\sin(\theta) = 4/6$. Also from the picture, we know that $\frac{\pi}{2} < \theta < \pi$, so $\theta \neq \sin^{-1}(4/6)$. (Recall that the range of $\sin^{-1}$ is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.) Hence θ must be the symmetry solution: $\theta = \pi - \sin^{-1}(4/6) \approx 2.412$ radians.

(c) Recall that the slope of this line must be $\tan(\theta)$, so $\tan(\theta) = 4$. But $\theta > \pi$, so again $\theta \neq \tan^{-1}(4)$. The angle $\tan^{-1}(4)$ is the angle between the positive x-axis and the line $y = 4x$, so this time $\theta = \pi + \tan^{-1}(4) \approx 4.467$ radians.

3 (a) To find $f^{-1}(x)$, we switch x and y and solve for y (essentially we're just solving for x in terms of y): $x = \frac{3y-1}{y-3}$. We get $x(y-3) = 3y-1$, so $yx - 3y = 3x - 1$, or $y = \frac{3x-1}{x-3}$. Hence $f^{-1}(x) = \frac{3x-1}{x-3}$ (which is, in fact, also $f(x)$.) The domain of this function is all real numbers *except* $x = 3$.

We do the same thing for $g(x) = x^2 + 1$. We get $x = y^2 + 1$, so $y = \pm\sqrt{x-1}$. To get a function, we need to choose one: let's take $g^{-1}(x) = +\sqrt{x-1}$. The domain of this function is real numbers $x \ge 1$.

(b) The vertical asymptote of the graph $y = f(x)$ is where the denominator $x - 3$ is zero but the numerator $3x - 1$ is not. This happens at $x = 3$, the vertical asymptote.

The horizontal asymptote is found by multiplying the top and bottom by $1/x$:

$$f(x) = \frac{3x-1}{x-3} \cdot \frac{1/x}{1/x} = \frac{3 - 1/x}{1 - 3/x}.$$

As x gets large (as a positive or negative number), $1/x$ gets close to zero, so $f(x)$ gets close to $3/1$. Thus the horizontal asymptote is $y = 3$.

(c) We have two wheels whose linear speeds are the same, and we are asked about or given information about their angular speeds. Thus we should probably be using the formula $v = r\omega$. To distinguish between the large and small wheel, I'll use a subscript L and S.

We are told that $\omega_L = 100$ RPM, $r_L = 8$ inches, and $r_S = 3$ inches. Also, we have that $v_L = v_S$. If both ω_L and ω_S are in *radians* per time unit, then we get

$$\begin{aligned} r_S\omega_S &= v_S && \text{(the basic formula)} \\ &= v_L && \text{(the linear speeds are the same)} \\ &= r_L\omega_L && \text{(the basic formula again).} \end{aligned}$$

Thus, $\omega_S = (r_L/r_S)\omega_L$ (in *any* units). Thus we have $\omega_S = \frac{8}{3}(100 \text{ RPM}) = \frac{800}{3}$ RPM ≈ 266.667 RPM.